MATHEMATICAL MORPHOLOGY "ALMOST EVERYWHERE"

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Abstract In this paper, we do not aim at new applications or algorithms, but at a formalism as simple and as practical as possible to deal with the several useful concepts in image and shape analysis, namely : level sets of a function, reconstruction of a function from its level sets, monotone set operators, contrast invariant monotone image operators, threshold superposition principle, sup-inf operators and flat morphology, image operators commuting with thresholds.

We prove that five slightly different terminologies or formalisms can be merged into a single simple presentation. Namely : the operators of "flat Mathematical Morphology", the "contrast invariant image operators", the "monotone set operators", the "sup-inf" operators and finally the "contrast invariant image operators defined on continuous images" are fully equivalent. In this equivalence statement, set functions are defined from set operators by the threshold superposition principle and set operators are defined from contrast invariant operators by the so-called Evans-Spruck extension. All that we prove may be known in different contexts but has not been formalized, to our knowledge, in a simple unified format. The closest theory to what we present, in Mathematical Morphology, is in the abstract framework of complete lattices. We do not request any completeness requirement in what follows and the statements apply to operators defined on any set of functions or any set of sets. As illustration, we show how the unified formalism permits to define easily several image operators by giving their simpler set operator definition and conversely how we also easily deal with set operators defined from P.D.E.'s, as it occurs with geodesic snakes. The formal presentation of contrast invariant mathematical morphology given here will be developed in the book [5] in project.

Keywords: level sets, monotone set operators, flat mathematical morphology, contrast-invariant operators, sup-inf operators, PDEs.

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1. Notations and summary of results

In the following, we denote by $\mathcal{F}$ a space of functions defined on $\mathbb{R}^N$ with values in $\mathbb{R}$ and denote by $\mathcal{T}$ the set of all level sets of functions of $\mathcal{F}$. ($\mathcal{F}$ will be interpreted as the set of the images). If (e.g.) $\mathcal{F}$ is the set of continuous real functions on $\mathbb{R}^N$, then $\mathcal{T}$ is the set of closed subsets of $\mathbb{R}^N$. If $\mathcal{F}$ is the set of Borel-Lebesgue measurable functions then $\mathcal{T}$ is the set of Borel subsets of $\mathbb{R}^N$, etc. We assume that $\mathcal{F}$ is stable by continuous non decreasing contrast changes $u \to g(u)$, i.e. $g(u) \in \mathcal{F}$ whenever $u \in \mathcal{F}$. We consider function operators on $\mathcal{F}$, $T : u \in \mathcal{F} \to Tu$, where Tu is a real function, $Tu(\mathbf{x}) \in \mathbb{R}$, and set operators on $\mathcal{T}$, $\mathtt{T} : X \in \mathcal{T} \to \mathtt{T}X$.

We say that a function operator T is monotone if $u \geq v \Rightarrow Tu \geq Tv$. We say that a set operator $\mathtt{T}$ is monotone if $X \subset Y \Rightarrow \mathtt{T}(X) \subset \mathtt{T}(Y)$. We call continuous contrast change any non decreasing continuous function $g : \mathbb{R} \to \mathbb{R}$. We say that T is contrast invariant on $\mathcal{F}$ if for every continuous contrast change $g : \mathbb{R} \to \mathbb{R}$, $\forall u \in \mathcal{F},\ g(Tu) = T(g(u))$.

If $u(\mathbf{x})$ is a real function, which we interpret again as an image, define the level set of u with level λ by $\mathcal{X}_\lambda u = \{\mathbf{x}, u(\mathbf{x}) \geq \lambda\}$, for $\lambda \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$). Obviously, $\mathcal{X}_{-\infty}u = \mathbb{R}^N$, $\mathcal{X}_{+\infty}u = \emptyset$. We can reconstruct u from its level sets $\mathcal{X}_\lambda u$ by the formula

$$u(\mathbf{x}) = \sup\{\lambda, \mathbf{x} \in \mathcal{X}_\lambda u\}.$$

Assume first that $\mathcal{F}$ contains all characteristic functions of elements of $\mathcal{T}$. Consider monotone set operators $\mathtt{T}$ (with $T(\emptyset) = \emptyset$, $T(\mathbb{R}^N) = \mathbb{R}^N$) and monotone contrast invariant function operators T. Then, we can either define T from $\mathtt{T}$, by the threshold superposition principle,

$$Tu(\mathbf{x}) = \sup\{\lambda,\ \mathbf{x} \in \mathtt{T}(\mathcal{X}_\lambda u)\} \tag{1}$$

or define $\mathtt{T}$ from T, provided the domain $\mathcal{F}$ of T contains the characteristic functions, by thresholding :

$$\mathtt{T}(X) = \mathcal{X}_1(T(\mathbb{1}_X)). \tag{2}$$

(We set $\mathbb{1}_X(\mathbf{x}) = 1$ if $\mathbf{x} \in X, = 0$ otherwise. As we shall prove, the relation between T and $\mathtt{T}$ is also characterized by the commutation with thresholds,

$$\mathtt{T}(\mathcal{X}_\lambda u) = \mathcal{X}_\lambda(Tu), \tag{3}$$

this relation being true almost everywhere in λ and $\mathbf{x}$. We shall deduce that every contrast invariant operator T has a sup inf formulation,

$$Tu(\mathbf{x}) = \sup_{B \in \mathbb{B}_{\mathbf{x}}} \inf_{\mathbf{y} \in B} u(\mathbf{y}). \tag{4}$$

for some set of sets $\mathbb{B}$. For the same $\mathbb{B}$, we also have the well known Matheron formula,

$$\mathtt{T}(X) = \bigcup_{B \in \mathbb{B}} \bigcap_{\mathbf{y} \in B} (X - \mathbf{y}).$$

$$\texttt{(1) T}\ \textbf{monotone,}\ \texttt{T}(X) = \bigcup_{B\in I\!B} \bigcap_{\mathbf{y}\in B} (X - \mathbf{y}).$$
$$\Uparrow$$
$$\texttt{T}(\mathcal{X}_\lambda u) = \mathcal{X}_\lambda(Tu) \text{ a.e. in } \lambda \text{ and } \mathbf{x}$$
$$\Downarrow$$
$$\texttt{(2)}\ T\ \textbf{monotone contrast invariant}$$
$$\Uparrow$$
$$I\!B = \{B, 0 \in \texttt{T}(B)\}$$
$$\Downarrow$$
$$\texttt{(3)}\ Tu = \mathbf{sup}_{I\!B}\mathbf{inf}_B(u)$$
$$\Downarrow$$
$$\texttt{(4)}\ T\ \textbf{contrast invariant on}\ C^0$$
$$\|$$
$$\texttt{T}(\mathcal{X}_\lambda u) = \mathcal{X}_\lambda(Tu)$$
$$\Downarrow$$
$$\texttt{(5) T}\ \textbf{defined on compact sets}$$
$$\|$$
$$\mathcal{X}_\lambda \tilde{T}u = \texttt{T}(\mathcal{X}_\lambda u),$$
$$\tilde{T} = T \text{ for all } u \in C^0$$
$$\Downarrow$$
$$\texttt{(2)}$$

Figure 1. Relations between set and function operators. (All operators are monotone and translation invariant - See text for exact assumptions). (1) $\Leftrightarrow$ (2): Propositions 8 and 9, (2) $\Rightarrow$ (3): Theorems 10 and 11, (3) $\Rightarrow$ (4): Theorem 11, (4) $\Rightarrow$ (5): Evans-Spruck Theorem, (5) $\Rightarrow$ (2): proved in [5].

All these formulations are well known and well described in the literature. We only propose here links between them based on equivalences almost everywhere (see correct definition later). In others words: *It is almost everywhere equivalent to define : a set monotone operator* T *with* $\texttt{T}(\emptyset) = \emptyset$, $\texttt{T}(I\!R^N) = I\!R^N$*, a contrast invariant image operator* T *or an* inf sup *operator. Each one of the forms of the operator is deduced from each other one with the help of Formulas (1)-(4) above.* This equivalence between the three modes of definition is summarized in the upper part of Figure 1.

Notice that the commutation with thresholds (3) is not even true for most dilations or erosions. (To take a simple instance: if we dilate a continuous function by an open ball, the commutation with threshold is not true- Indeed, the dilate of the function is still a continuous function, with closed level sets. The dilates of the level sets of the function are instead all open). The *commutation is instead true almost everywhere* and we shall prove it and use it extensively.

2. Main References

On this subject, the books of Matheron [18] (concerning sets) and Serra [21], Serra[22] are seminal. In "flat morphology", images are decomposed into their level sets by the so-called threshold decomposition so that image analysis boils down to set analysis. Book of Heijmans [8] proposes some more general extensions and gives a strong background on the subject.

Image operators commuting with thresholds have been very popular because of their simplicity of implementation on VLSI, which led to very simple patents in image or signal processing as late as 1987 [3]. These operators have received four different but equivalent names : "stack filters" [25, 6, 1], "threshold decomposition" [10], "rank filters" [11] and "order filters" [23]. The most famous such operators are the sup, inf and median operators. The implementation of the last one has received a lot of attention because of his remarkable denoising properties [26, 19].

The papers by Maragos and Shafer [15, 14] and Maragos and Ziff [16] introduce the functional notation in the debate and establish the link between stack filters and the Matheron formalism in "flat" mathematical morphology. Actually, Maragos and his collaborators prove the equivalence between stack filters and operators commuting with thresholds. A strong theoretical study has been done by Heijmans [8]. The full equivalence between contrast invariant operators and stack filters, proved in this article, does not seem instead to be classical. A related classification of rank filters with beautiful and useful generalizations to the so-called "neighborhood filters" can be found in [11].

The formalism presented in section 4.3 is due to Matheron [18] in the case of the set operators and to Serra [21] and Maragos [13] in the case of function operators. Maragos proved the sup-inf formula given here for operators commuting with thresholds. Our presentation is original in relating directly the sup-inf form to contrast invariance and establishing the full equivalence between sup-inf operators and contrast invariant monotone operators.

The proof and use of relation (3), true "almost everywhere" is essential in establishing the announced links.

3. From an image to its level sets, and conversely.

We say that a set X is contained in a set Y almost everywhere if $meas(X \setminus Y) = 0$, where $meas$ denotes the usual Lebesgue measure. We say that $X = Y$ almost everywhere if $X \subset Y$ and $Y \subset X$ almost everywhere. We say that two functions u and v are almost everywhere equal if $meas(\{\mathbf{x}, u(\mathbf{x}) \neq v(\mathbf{x})\}) = 0$. More generally, we say that a property $P(\lambda)$, $\lambda \in \mathbb{R}^N$, is true " almost everywhere" or "for almost every λ" if it is true for every λ, with the exception of a set with zero N-dimensional Lebesgue measure. We also set $\inf(\emptyset) = +\infty, \quad \sup(\emptyset) = -\infty$.. Let us state the following lemma whose proof can be found in any book on measure theory.

Lemma 1 *Let $(X_\lambda)_{\lambda \in \mathbb{R}}$ be a non increasing family of sets, i.e. $X_\lambda \subset X_\mu$ if $\lambda \geq \mu$. Then, for almost every λ in $\mathbb{R}$,*

$$X_\lambda = \bigcap_{\mu < \lambda} X_\mu, \qquad \textit{almost everywhere} \tag{5}$$

Proposition 2 *Let $(X_\lambda)_{\lambda \in \overline{\mathbb{R}}}$ a family of subsets of $\mathbb{R}^N$ such that $X_{-\infty} = \mathbb{R}^N$, $X_\lambda \subset X_\mu$ for $\lambda \geq \mu$. Then the function u defined by*

$$u(\mathbf{x}) = \sup\{\lambda, \mathbf{x} \in X_\lambda\}$$

satisfies for almost every λ, $X_\lambda = \mathcal{X}_\lambda u$ almost everywhere. This function has values in $\overline{\mathbb{R}}$.

Proof. We have $\mathcal{X}_\lambda u = \{\mathbf{x}, \sup\{\mu, \mathbf{x} \in X_\mu\} \geq \lambda\}$ Now, if $\mathbf{x} \in X_\lambda$, we have $\sup\{\mu, \mathbf{x} \in X_\mu\} \geq \lambda$ which implies $\mathbf{x} \in \mathcal{X}_\lambda u$. Thus, $X_\lambda \subset \mathcal{X}_\lambda u$ everywhere. Conversely, let λ be chosen such that $X_\lambda = \cap_{\mu < \lambda} X_\mu$ almost everywhere, which by Lemma 1 is true for almost every $\lambda \in I\!R$. Then if $\mathbf{x} \in \mathcal{X}_\lambda u$, we have by definition of u, $\mathbf{x} \in X_\mu$ for every $\mu < \lambda$. Thus $\mathbf{x} \in \bigcap_{\mu<\lambda} X_\mu$. We conclude that $X_\lambda u \subset \bigcap_{\mu<\lambda} X_\mu$ and therefore $\mathcal{X}_\lambda u \subset X_\lambda$ almost everywhere. ◦

Let us state a lemma ensuring that if we know almost everywhere the level sets of a function for almost all levels, then the function itself can be retrieved, up to a set with measure zero.

Lemma 3 *Let v be a function and $(Y_\lambda)_{\lambda \in I\!R}$ be a family of sets such that $\mathcal{X}_\lambda v = Y_\lambda$, a.e. in λ, a.e. in $\mathbf{x}$. Then $v(x) = \sup\{\lambda,\ \mathbf{x} \in Y_\lambda,\}$ a.e. in $\mathbf{x}$.*

Proof. Let N be the negligible subset of $I\!R$ such that $\mathcal{X}_\lambda u = Y_\lambda$ almost everywhere and for all $\lambda \in I\!R \setminus N$. We choose $\Lambda \subset I\!R \setminus N$, a countable, dense subset of $I\!R$. We then still have $v(\mathbf{x}) = \sup\{\lambda \in \Lambda,\ u(\mathbf{x}) \in \mathcal{X}_\lambda v\}$. Let now $N_\lambda = (\mathcal{X}_\lambda u \setminus Y_\lambda) \cup (Y_\lambda \setminus \mathcal{X}_\lambda u)$ for $\lambda \in \Lambda$ and $M = \bigcup_{\lambda \in \Lambda} N_\lambda$. We have $meas(M) = 0$ and for $\mathbf{x} \in I\!R^N \setminus M$, $v(\mathbf{x}) = \sup\{\lambda \in \Lambda,\ \mathbf{x} \in \mathcal{X}_\lambda v,\ \} = \sup\{\lambda \in \Lambda,\ \mathbf{x} \in Y_\lambda.\}$ ◦

We conclude this section with a simple to remember statement (proved in [5]): if all level sets of v are level sets of of u, then $v = g(u)$ for some g. In other words "image modulo contrast = level sets". The precise statement is

Corollary 4 *If v and u are two real functions such that all level sets of v are level sets of u, then there is a non decreasing function $g : \overline{I\!R} \to \overline{I\!R}$ such that $v = g(u)$. An instance of such a function g is*

$$g(\mu) = \sup\{\lambda, \mathcal{X}_\lambda v \supset \mathcal{X}_\mu u\}. \tag{6}$$

4. Set operators and contrast invariant operators.

4.1 From contrast invariant operators to set operators : the threshold superposition principle.

Assume that $\mathcal{F}$ contains the characteristic functions $1\!\!1_X$ of elements of X. We can associate with T a set operator $\mathtt{T}$, defined on the set $\mathcal{T}$ of all level sets of all functions in $\mathcal{F}$ by

$$\mathtt{T}(X) = \mathcal{X}_1(T(1\!\!1_X)). \tag{7}$$

where $1\!\!1_X(\mathbf{x}) = 1$ if $\mathbf{x} \in X$ and $1\!\!1_X(\mathbf{x}) = 0$ otherwise. Note that if T is monotone, then $\mathtt{T}$ is a set monotone operator. The definition of $\mathtt{T}$ makes sense by the following proposition : according to it, $T(1\!\!1_X)$ attains the values 0 and 1 like u and is therefore a characteristic function.

Proposition 5 *[5] Let T be a contrast invariant operator. If u attains a finite number of values, then Tu attains a subset of them.*

Lemma 6 *Let T be a contrast invariant monotone operator. Consider the threshold functions $\gamma_\lambda(s) = 1$ if $s \geq \lambda$ and $\gamma_\lambda(s) = 0$ otherwise. Then T commutes almost everywhere with almost every threshold, i.e.*

$$\gamma_\lambda(Tu) = T(\gamma_\lambda(u)) \text{ a.e. in } \lambda, \quad \text{a.e. in } \mathbf{x}.$$

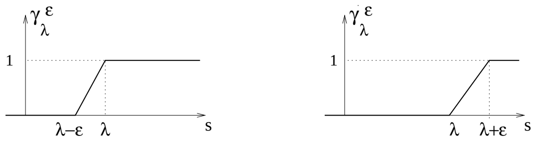

Figure 2. Approximation of the threshold function γ_λ from above and from below

Proof. We consider $\gamma_\lambda^\epsilon(s)$, as defined in figure 2. $\gamma_\lambda^\epsilon(s)$ is a contrast change and $\gamma_\lambda^\epsilon \geq \gamma_\lambda$. Thus

$$T(\gamma_\lambda(u)) \leq T(\gamma_\lambda^\epsilon(u)) = \gamma_\lambda^\epsilon(Tu) \to \gamma_\lambda(Tu), \quad \text{as } \epsilon \to 0.$$

Using in the same way continuous non decreasing functions $\tilde{\gamma}_\lambda^\epsilon \leq \gamma_\lambda$, we also prove that $T(\gamma_\lambda(u)) \geq \gamma_\lambda^-(Tu)$, where $\gamma_\lambda^-(s) = 1$ if $s > \lambda$ and $\gamma(s) = 0$ otherwise. We therefore obtain $\gamma_\lambda^-(Tu) \leq T(\gamma_\lambda(u)) \leq \gamma_\lambda(Tu)$. Let us consider the countable, and therefore negligible, subset $\Lambda \subset I\!R$ of all λ such that $meas(\{\mathbf{x}, Tu(\mathbf{x}) = \lambda\}) > 0$. For $\lambda \in I\!R \setminus \Lambda$, we have $\gamma_\lambda^-(Tu) = \gamma_\lambda(Tu)$ almost everywhere. ∘

In a converse way to Relation (7) defining a set operator from a function operator, we can define a function operator T from a set operator $\mathtt{T}$ thanks to the *threshold superposition principle.*

Definition 7 *We say that T is obtained from $\mathtt{T}$ by the threshold superposition principle almost everywhere if*

$$Tu(\mathbf{x}) = \sup\{\lambda,\ \mathbf{x} \in \mathtt{T}(\mathcal{X}_\lambda u)\},\ \textit{a.e. in } \mathbf{x}. \tag{8}$$

Remark: For digital quantized images, equality a.e. is equivalent to equality everywhere!

Proposition 8 *Let T be a monotone contrast invariant operator on a set of functions $\mathcal{F}$ containing the characteristic functions $1\!\!1_X$ of the elements X of $\mathcal{T}$. Define its associated set operator by $\mathtt{T}(X) = \mathcal{X}_1(T(1\!\!1_X))$. Then T is monotone, we have for every $u \in \mathcal{F}$*

$$\mathtt{T}(\mathcal{X}_\lambda u) = \mathcal{X}_\lambda(T(u)),\ \textit{a.e. in } \lambda,\quad \textit{a.e. in } \mathbf{x} \tag{9}$$

and the threshold superposition principle holds almost everywhere :

$$Tu(\mathbf{x}) = \sup\{\lambda,\ \mathbf{x} \in \mathtt{T}(\mathcal{X}_\lambda u)\},\ \textit{a.e. in } \mathbf{x}. \tag{10}$$

In addition,

$$\mathtt{T}(\emptyset) = \emptyset\ \textit{a.e.},\ \mathtt{T}(I\!R^N) = I\!R^N\ \textit{a.e.}.$$

We so have the following "stack filter" algorithm:

$$u \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{ccc} \mathcal{X}_\lambda u & \to & \mathtt{T}(\mathcal{X}_\lambda u) \\ \cdots & & \\ \mathcal{X}_\mu u & \to & \mathtt{T}(\mathcal{X}_\mu u) \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \begin{array}{c} Tu(\mathbf{x}) = \\ \sup\{\lambda, \mathbf{x} \in \mathtt{T}(\mathcal{X}_\lambda u)\}. \\ a.e. \end{array}$$

Proof. Using the definition of $\mathtt{T}$, the obvious relations $1\!\!1_{\mathcal{X}_\lambda u} = \gamma_\lambda(u)$, $\mathcal{X}_1(\gamma_\lambda(v)) = \mathcal{X}_\lambda v$, and the commutation almost everywhere of T with γ_λ, obtained in Lemma 6, we get $\mathtt{T}(\mathcal{X}_\lambda u) = \mathcal{X}_1(T(1\!\!1_{\mathcal{X}_\lambda u})) = \mathcal{X}_1(T(\gamma_\lambda(u))) = \mathcal{X}_1(\gamma_\lambda(Tu)) = \mathcal{X}_\lambda(Tu)$, a.e. in λ, a.e. in $\mathbf{x}$. The superposition principle (8) follows immediately from (9) and Lemma 3, applied to $Y_\lambda = \mathtt{T}(\mathcal{X}_\lambda u)$ and $v = Tu$. ○

4.2 From set operators to contrast invariant function operators.

Proposition 9 *Let* $\mathtt{T}$*,* $\mathcal{T} \to \mathcal{T}$ *be a monotone operator satisfying* $T(\emptyset) = \emptyset, \mathtt{T}(I\!R^N) = I\!R^N$*, and being defined on* $\mathcal{F}$ *by superposition principle (1).* T *satisfies, for almost every* $\lambda \in I\!R$*,*

$$\mathcal{X}_\lambda(Tu) = \mathtt{T}(\mathcal{X}_\lambda(u)) \text{ almost everywhere in } \mathbf{x} \tag{11}$$

and for any non decreasing continuous contrast change g*,* $g(Tu) = T(g(u))$*.*

Proof. Corollary 2 yields almost immediately (11). Let us only show that T commutes with contrast changes. Assume first that g is strictly increasing and set $g(+\infty) = \lim_{s\to+\infty} g(s)$ and $g(-\infty) = \lim_{s\to-\infty} g(s)$.
For $\lambda > g(+\infty)$, we have $\mathcal{X}_\lambda g(u) = \emptyset$ and therefore $\mathtt{T}(\mathcal{X}_\lambda g(u)) = \emptyset$.
For $\lambda < g(-\infty)$, we have $\mathcal{X}_\lambda g(u) = I\!R^N$ and therefore $\mathtt{T}(\mathcal{X}_\lambda g(u)) = I\!R^N$.
So, using (1) we can restrict the range of λ in the definition of $T(g(u(\mathbf{x})))$:

$$T(g(u(\mathbf{x}))) = \sup\{\lambda,\ g(-\infty) \le \lambda \le g(+\infty),\ \mathbf{x} \in \mathtt{T}(\mathcal{X}_\lambda g(u))\}$$

$$= \sup\{g(\mu),\ \mathbf{x} \in \mathtt{T}(\mathcal{X}_{g(\mu)} g(u))\} = \sup\{g(\mu),\ \mathbf{x} \in \mathtt{T}(\mathcal{X}_\mu u)\} = g(Tu(\mathbf{x})).$$

Let us now check that T commutes with general non decreasing contrast changes g. We can find strictly increasing continuous functions g_n and h_n such that $g_n(s) \to g(s)$, $h_n(s) \to g(s)$ for all s and $g_n \le g \le h_n$. Thus, by using the just proven commutation of T with increasing contrast changes, we have $T(g(u)) \ge T(g_n(u)) = g_n(Tu) \to g(Tu)$ and , and the other inequality with h_n, yielding the result. ○

4.3 Sup-inf operators and monotone set operators.

Theorem 10 (Matheron) *Let* $\mathtt{T}$ *be a translation invariant monotone operator acting on a set of subsets of* $I\!R^N$*. Then, there exists a family of sets* $I\!B \subset \mathcal{P}(I\!R^N)$*, which can be defined as* $I\!B = \{X, 0 \in \mathtt{T}(X)\}$*, such that*

$$\mathtt{T}(X) = \bigcup_{B \in I\!B} \bigcap_{\mathbf{y} \in B} (X - \mathbf{y}) = \{\mathbf{x},\ \exists B \in I\!B,\ \mathbf{x} + B \subset X\}. \tag{12}$$

Conversely, (12) defines a monotone, translation invariant operator on $\mathcal{P}(I\!R^N)$*.*

Theorem 11 *(Variant of the preceding theorem) Let* $\mathcal{F}$ *a set of functions,* $\mathcal{T}$ *the set of all level sets of functions of* $\mathcal{F}$*. Assume that* $\mathcal{F}$ *is stable under* *contrast changes and contains the characteristic functions* $\mathbb{1}_X$ *of elements of* $\mathcal{T}$*. Let* T *be a monotone, contrast and translation invariant operator on* $\mathcal{F}$*. Let* $\mathtt{T}$ *be the set operator associated with* T*,* $\mathtt{T}(X) = \mathcal{X}_1(T(\mathbb{1}_X))$*. Then setting* $\mathbb{B} = \mathbb{B}_0 = \{X, 0 \in \mathtt{T}(X)\}$ *we have*

$$Tu(\mathbf{x}) = \sup_{B \in \mathbf{x} + \mathbb{B}} \inf_{\mathbf{y} \in B} u(\mathbf{y}),\ \textit{a.e. in } \mathbf{x} \tag{13}$$

Conversely, if an operator is defined by (13), then it is monotone and contrast invariant.

Proof of Theorem 11. Set $\tilde{T}u(\mathbf{x}) = \sup_{B \in \mathbb{B}} \inf_{\mathbf{y} \in B} u(\mathbf{y})$. We argue as in Lemma 3 : We choose a countable dense set $\Lambda \subset \mathbb{R}$ such that for every $\lambda \in \Lambda$, $\mathcal{X}_\lambda Tu(\mathbf{x}) = \mathtt{T}(\mathcal{X}_\lambda u)(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^N \setminus N_\lambda$, where $meas(N_\lambda) = 0$. We set $N = \bigcup_\lambda N_\lambda$ and we still have $meas(N) = 0$. Proof comes by establishing that for all $\lambda \in \Lambda$ and all $\mathbf{x} \in \mathbb{R}^N \setminus N$, we have $\tilde{T}u(\mathbf{x}) \geq \lambda \Leftrightarrow Tu(\mathbf{x}) \geq \lambda$. and then applying Lemma 3.

5. Applications

The preceding sections provide an easy framework to switch between set operators, sup-inf image operators and contrast invariant image operators.

Median. Weighted Median Filtercan be defined as an "sup-inf" filter as follows: Let for any set B, $|B|_k = \int_{\mathbf{x} \in B} k(\mathbf{x}) d\mathbf{x}$ where k the positive radial function defining the weight. (For the median filter on a disk of radius r: choose $k(\mathbf{x}) = 1/(\pi r^2)$ if $\mathbf{x}$ is on the disk and 0 otherwise). We call weighted median set of X the set:

$$med_k(X) = \{\mathbf{x}; |X - \mathbf{x}|_k \geq \frac{1}{2}\}.$$

(The median filter applied on a set X can be seen as a "majority" filter. It yields the points whose neighborhood is majority made of points of X.)

Applying in turn Proposition 9 and Theorem 11 we get immediately a definition of the weighted median image filter in "sup-inf" form, as a contrast invariant operator and as a stack filter (ie: satisfying the superposition principle). $\mathrm{med}_k(u)(\mathbf{x}) = \sup_{B; |B|_k \geq \frac{1}{2}} \inf_{\mathbf{y} \in B + \mathbf{x}} u(\mathbf{y})$. When k is a Gaussian function, its associated set operator is nothing but the so called "dynamic shape" method [12] for shape analysis.

Level set methods and active contour. Now, we shall also have cases, namely the case of solutions of partial differential equations, where the operator is much easier to define as an operator on a set of continuous functions. This is the case for the solutions of the curvature motion, the affine morphological scale space or active contour models. Active contours are evolved by modifying their signed distance function. Evolution of the distance function is generaly described by a parabolic Partial Differential Equation. The evolved active contour is defined as the boundary of the zero-level set of the evolved function by the PDE.

The operator T which associates to the initial distance function the solution (in viscosity sense) at some given scale is monotone and contrast invariant on the set of Lipschitz functions for most proposed models. Evans-Spruck method described in [4] applies and we directly get an extension of the operator T as a compact set operator $\mathtt{T}$ by Relation (3), being then true everywhere (for all λ and all $\mathbf{x}$). The threshold decomposition principle ensures that the zero level set (and therefore the active contour) of the solution of the PDE, moves according to this set operator.

More Generally, given a contrast invariant operator T defined on continuous functions, we can define an extension $\tilde{T}$ which satisfies all of equivalence properties of the lower part of Figure 1. This situation is again summarized in the lower part of the figure 1: from the contrast invariance on continuous functions, we go down to define a set operator on compact sets by thresholding and then a function operator on a more general set of functions.

Extrema killer - or Area Opening/Closing. Extrema killer is a typical example of an operator easy to define on sets but rather complex and uncomputable as an sup-inf operator on images... A first attempt of this filter seems to be [2]. The filter in its generality was defined in Vincent [24]. Its definition fits in the general theory of connected filters developed by Salembier and Serra [20]. Note the variant introduced by Masnou [17] that is invariant under reverse contrast changes.

Fix a scale parameter $a > 0$. Let X be a compact set. Then X is the union of all of its connected components, $X = \bigcup_i X_i$ and this decomposition is uniquely defined. Remove from X all connected components of measure (area in 2D) strictly less than a : We so define a "small component killer" operator:

$$\mathtt{T}_a(X) = \bigcup_{meas(X_i) \geq a} X_i. \tag{14}$$

It is easily seen that $\mathtt{T}_a$ is monotone.

One can associate with $\mathtt{T}_a$ a "maxima killer", T_a, defined on continuous functions u. This operator T_a is defined by the threshold decomposition principle (1). So that we have: $\mathcal{X}(T_a u) = \mathtt{T}_a(\mathcal{X}u)$ and, as a consequence, no connected maximum set of $T_a u$ has area less than a. In addition, $T_a u$ is continuous.

T_a can stand as "maxima killer" and we could define in the same way a "minima killer". A good denoiser can be obtained by alternating T_a and T_a^-. This alternating is not illicit since each one of both extrema killers maintains the continuity of the image.

6. Conclusion

Monotone set operators in $I\!R^N$, Sup-inf operators and contrast invariant functions operators are strictly equivalent classes. We can use the easiest way to define or implement such operators. We have illustrated this flexibility by three classical examples. In the book [5] we use this fact to classify all monotone set operators, understood as the widest class shape smoothing operators.

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