PARTIAL ORDERINGS AND SCALE-SPACES WITH MONOTONICITY OF FEATURES

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Abstract Signal simplification, as measured through a reduction in signal features is an important aspect of scale-space theories. *Partial ordering* can be seen as a formal characterization of the notion of a simpler version of a signal. The partial orderings implied by some scale-spaces which obey monotonicity of features are examined with examples drawn from 1-D Gaussian scale-space and Multiscale Dilation Scale-Space. We hope to provide a unifying framework for such scale-spaces

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1. Introduction

As we have observed previously, if scale-space is to be something more useful than a stack of signal descriptions we must demand that the signal representation gets simpler with increasing scale [1]. This is usually done through imposing monotonicity conditions on signal features (*wrt* scale), although as Witkin pointed out, "some means must be found to organize the description by relating one scale to another" [2].

Some recent research has provided a general algebraic framework for scale-space operators which can unify linear and non-linear (morphological and PDE) approaches [3, 4]. This work has tended (not-surprisingly!) to emphasize the algebraic (semigroup) properties of scale-space and to place less emphasis on signal simplification. How then should we tackle the signal simplification issues? Recent insights by Keshet may provide a clue: "the concept of *partial ordering* could be seen as a formal characterization of the notion of a simpler version of a signal" [5].

In this paper we explore the partial orderings implied by some scale-spaces which obey monotonicity of features. We hope to provide a unifying framework for such scale-spaces.

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2. Partial Orderings

In the context of *adjunctions* and extending theoretical morphology from complete lattices through complete semi-lattices and on to generic partially ordered sets (posets), Keshet has unified and formalized the notions of signal information content, signal simplification and reconstruction, and filtering, through properly chosen partial orderings on the appropriate lattices and sets [5, 6]. The major definitions are:

Definition 1 (Partial Ordering) *A relation $\leq$ in a set $\mathcal{S}$ is a partial ordering if it is reflexive ($s \leq s$), anti-symmetric (if $s \leq r$ and $r \leq s$, then $s = r$), and transitive (if $s \leq r$ and $r \leq q$, then $s \leq q$).*

A set ordered by a partial ordering is called a partially ordered set or *poset*.

Definition 2 (Adjunctions) *Let $\leq$ be a partial ordering defined on a set $\mathcal{S}$. Let $\epsilon : \mathcal{S} \mapsto \mathcal{S}$ and $\delta : \mathcal{S} \mapsto \mathcal{S}$ be two operators. The pair (ϵ, δ) is called an adjunction on the poset $(\mathcal{S}, \leq)$ if $r \leq \epsilon(s) \Leftrightarrow \delta(r) \leq s, \quad \forall s, r \in \mathcal{S}$.*

Following [7] we can define morphological erosions and dilations (and openings and closings) in terms of adjunctions.

Definition 3 (Erosions and Dilations) *An operator $\epsilon : \mathcal{S} \mapsto \mathcal{S}$ is called an erosion on poset $(\mathcal{S}, \leq)$ iff there exists an operator $\delta : \mathcal{S} \mapsto \mathcal{S}$ called a dilation such that (ϵ, δ) is an adjunction on the poset $(\mathcal{S}, \leq)$.*

Note: the dilation which forms an adjunction with a given erosion is unique (and *vice versa*) [7, Proposition 1]. The erosion ϵ and dilation δ are both increasing, that is, $s \leq r \Rightarrow \epsilon(s) \leq \epsilon(r)$ and $\delta(s) \leq \delta(r)$ [7, Proposition 2].

Definition 4 (Openings and Closings) *Let (ϵ, δ) be an adjunction, then $\alpha = \delta\epsilon$ is called the opening, and $\beta = \epsilon\delta$ is called the closing associated with the adjunction.*

3. Scale-spaces

Van den Boomgaard & Heijmans have recently presented an algebraic framework for scale space operators [3]. One of the many good ideas in this approach is to explicitly separate out the scaling and rescaling operators from the actual image operator, that is if the scale-space operator is denoted by T_t, where $t > 0$ is the scale parameter, then we let

$$T_t = S_t \, \psi S_t^{-1} \tag{1}$$

where S_t is a scaling operator defined as follows.

Definition 5 (Scaling) *Let $\mathcal{L} = Fun(R^d, \bar{R})$ be the space of images. A family $S = \{S_t | t > 0\}$ of operators on $\mathcal{L}$ is called a scaling if (i) $S_1 = \mathbf{id}$ and (ii) $S_t S_s = S_{ts}$ for $s, t > 0$.*

Note that, $S_t^{-1} = S_{1/t}$. Under this scheme the image is down-scaled by factor t, an operator is applied at unit scale, and the image is up-scaled by factor t to its original size. Van den Boomgaard & Heijmans are then concerned with the semigroup properties of compositions of scale-space operators, whereas we will head in a different direction by adding in the notion of features to the formulation. To do this let's formalize the idea of feature extraction through an appropriate operator.

A signal feature or image feature (however defined) occurs at a definite set of positions in the domain of the signal. Previously[1], we have used set notation for this concept but now, to better fit the above framework, we will use an indicator image approach.

Definition 6 (Feature extraction) *Let $\mathcal{L} = Fun(R^d, \bar{R})$ be the space of images, and $f(x) \in \mathcal{L}$ be such an image. An operator $\xi : \mathcal{L} \mapsto \mathcal{L}$ on is called a feature extractor if* $(\xi f)(x) = \begin{cases} 1 & \textit{if feature occurs at } f(x); \\ 0 & \textit{otherwise.} \end{cases}$

Now we can build our scale-space feature extraction operator X_t,

$$X_t = S_t \, \xi S_t^{-1} \qquad (2)$$

where S_t is the scaling operator defined previously. To extract features at a fixed scale t we first apply the scale-space operator T_t followed by the feature extraction operator X_t:

$$\begin{aligned} X_t T_t &= S_t \, \xi S_t^{-1} S_t \psi S_t^{-1} & (3) \\ &= S_t \, \xi \psi S_t^{-1}. & (4) \end{aligned}$$

This operator is used to extract a feature indicator image from the scale-space image,

$$\begin{aligned} F_t(x) &= X_t \, T_t \, f(x) & (5) \\ &= X_t \, f_t(x), & (6) \end{aligned}$$

where the scale-space image $f_t(x) = T_t \, f(x)$.

A scale-space "fingerprint" is a plot of signal feature positions versus scale [8] Extracting features from scale space to build up a fingerprint would therefore consist in principle of applying the scale-space operator to the image at scale zero followed by the feature extraction operator at the same scale scale, and repeating for all scales $t > 0$ of interest. As equation 4 shows, we can also regard the feature extraction operator as being applied at unit scale following the ψ operator, and before rescaling.

The monotonicity condition for features requires that feature paths be continuous in scale-space so that each feature may be tracked back through scale-space along this path to its unique position on the original image. Although this may actually present some difficulties in practice due to the discrete sampling of scale, we will assume that this can be done in principle and we will express the monotonicity property for features as:

Property 1 (Monotonicity for Features) *Let $f(x)$ be any signal and let its feature indicator image at scale t, be extracted according to equation 5. Then for all feature indicator images of lesser scales, $F_s(x) = X_s\, T_s\, f(x) \quad 0 \leq s < t$, each feature point in $F_t(x)$ can be uniquely identified with a feature point in $F_s(x)$.*

The leads naturally to the following partial ordering for feature indicator images and we define:

Definition 7 (Partial Ordering for Feature Indicator Images) *Let F_t and F_s with $0 \leq s < t$ be feature indicator images obeying the Monotonicity property. Then we write $F_t \preceq F_s$. This relation is one of partial ordering.*

Since equation 6 shows each feature indicator image F_t to be uniquely associated with a scale-space image f_t (through operator X_t) we can define a partial order on the scale space images themselves.

Definition 8 (Partial Ordering for Scale-Space Images) *Let $F_t = X_t\, f_t$ and $F_s = X_s\, f_s$ and $F_t \preceq F_s$. Then we write $f_t \preceq f_s$. This relation is also one of partial ordering.*

Note we use the notation $\preceq$ as a reminder that this is not the standard ordering for images.

Now, the problem of forming scale-spaces with monotonicity of features becomes one of finding operators which give appropriate orderings. Some examples are given in the next section.

4. Examples of Scale-Spaces with Monotonicity of Features

4.1 1-D Gaussian Scale-Space

Our first example must surely be the ubiquitous 1-D Gaussian scale-space with the Marr-Hildreth edge detector (zero-crossings of second derivative) features as described by Witkin and others in the 1980's [2, 8, 9](although independently and previously in Japan in the 1950's by Iijima *et. al.*, see [10].)

Let $f(x) : R \mapsto R$ denote a signal and consider the 1-D Gaussian Scale-Space generated by

$$(T_t f)(x) = (2\pi t)^{-1/2} \int f(x-y)\, \exp\left(-\frac{y^2}{2t}\right) dy. \tag{7}$$

We can easily see that from equation 1 (or from [3]) that the image operator is the convolution with the unit Gaussian,

$$(\psi f)(x) = (2\pi)^{-1/2} \int f(x-y)\, \exp\left(-\frac{y^2}{2}\right) dy, \tag{8}$$

and the scaling is,

$$S_t f = f(\cdot/\sqrt{t}). \tag{9}$$

The feature extraction operator is,

$$(\xi f)(x) = \begin{cases} 1 & \text{if } \frac{d^2}{dx^2} f(x) = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The fingerprint theorems which guarantee continuity and monotonicity for these features [8] in turn guarantee the partial ordering as proposed above with these operators.

4.2 Multiscale Dilation Scale-Space

Our second example is the multiscale dilation scale-space we have previously proposed [1]. That approach "joined" an erosion scale-space to a dilation scale-space at zero scale, for present purposes we will just take one side of this, say the dilation scale-space where features are signal maxima.

Let $f(x) : R^2 \mapsto R$ denote an image and consider the Scale-Space generated by

$$(T_t f)(x) = f(x) \oplus g_t(x), \tag{11}$$

where, $\oplus$ is the morphological dilation and $g_t(x)$ is the "scaled poweroid" structuring function

$$g_t(x) = -t(||x||/t)^\alpha \tag{12}$$

The image operator is the dilation with the unit structuring function,

$$(\psi f)(x) = f(x) \oplus -||x||^\alpha, \tag{13}$$

and the scaling is,

$$S_t f = t\, f(./t). \tag{14}$$

The feature extraction operator is,

$$(\xi f)(x) = \begin{cases} 1 & \text{if } x \text{ is a local maximum of } f(x); \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

The fingerprint theorems which guarantee continuity and monotonicity for these features [1] in turn guarantee the partial ordering as proposed above with these operators.

5. Conclusion

We have explored the partial orderings implied by some scale-spaces which formally obey monotonicity of features. We have been able to place two classical scale-spaces, one linear — one not, in the proposed unifying framework. The idea of forming adjuctions (and hence an embedded scaled morphology) with respect to these operators is left to future work.

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