VISCOUS LATTICES

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Abstract Let E be an arbitrary space, and δ a dilation of $\mathcal{P}(E)$ into itself, with an adjoint erosion ε. Then, the image $\delta[\mathcal{P}(E)]$ of $\mathcal{P}(E)$ by δ is a complete lattice $\mathcal{L}$ where the sup is the union and the inf the opening of the intersection according to $\delta\varepsilon$. The lattice $\mathcal{L}$, named viscous, is not distributive, nor complemented. Any dilation α on $\mathcal{P}(E)$ admits the same expression in $\mathcal{L}$. However, the erosion in $\mathcal{L}$ is the opening according to $\delta\varepsilon$ of the erosion in $\mathcal{P}(E)$. The image under δ of any connection $\mathcal{C}$ on $\mathcal{P}(E)$ is a connection $\mathcal{C}'$ on $\mathcal{L}$. Moreover, if $\delta(\mathcal{C}) \subseteq \mathcal{C}$, then the elementary connected openings γ_x of $\mathcal{C}$ and $\gamma'_{\delta(x)}$ are linked by the relation $\gamma'_{\delta(x)} = \delta\varepsilon\ \gamma_x$. Two examples, binary and numerical (this latest one comes from the heart imaging), prove the relevance of viscous lattices in interpolation problems.

Keywords: Connection, interpolation, non-distributive lattices, geodesy.

1. Viscous Lattices

The present study is a comment on a paper of C. Vachier and F. Meyer [7], who propose to regularize a watershed by so-called *viscous propagations.* Their algorithm suggest to replace the usual working lattice $\mathcal{P}(E)$ by the more convenient framework that we develop below.

Let E be an arbitrary set, $\mathcal{P}(E)$ the lattice of its subsets, and let δ be a dilation : $\mathcal{P}(E) \to \mathcal{P}(E)$, of adjoint erosion ε. We know that δ is determined by the images of singletons $\{x\}$ of $\mathcal{P}(E)$, since the dilate $\delta(X)$ of any $X \in \mathcal{P}(E)$ is the union of the dilates of the singletons it contains :

$$\delta(X) = \cup\{\delta(x), x \in X\} \qquad X \in P(E) \tag{1}$$

Proposition 1 *The family $\mathcal{L} = \{\delta(X), X \in \mathcal{P}(E)\}$ is both the image of $\mathcal{P}(E)$ under dilation δ and under the opening $\gamma = \delta\varepsilon$, adjoint to the dilation δ.*

Proof. *If $X \in \mathcal{P}(E)$ is open according to $\delta\varepsilon$, then*

$$X = \delta\varepsilon(X) = \delta(Y) \qquad \text{with } Y = \varepsilon(X).$$

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Conversely, if $X = \delta(Y)$*, then* $\delta\varepsilon(X) = \delta\varepsilon\delta(Y) \supseteq \delta(Y) = X$ *by extensivity of the closing* $\varepsilon\delta$*, and also* $\delta\varepsilon(X) \subseteq X$ *by anti-extensivity of the opening* $\delta\varepsilon$ *; therefore,* X *is opened by* $\delta\varepsilon$*.* ■

Moreover, we have the classical following result :

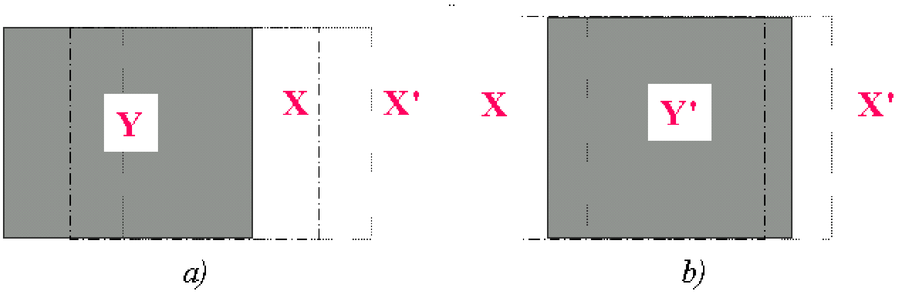
Proposition 2 *The set* $\mathcal{L}$ *is a complete lattice regarding the inclusion ordering. In this lattice, the supremum coincides the set union, whereas the infimum* $\wedge$ *is the opening according to* $\gamma = \delta\varepsilon$ *of the intersection.*

$$\wedge\{X_i, i \in I\} = \gamma(\cap\{X_{i,}i \in I\}) \qquad\qquad \{X_i, i \in I\} \in \mathcal{L} \tag{2}$$

The extreme elements of $\mathcal{L}$ *are* E *and the empty set* $\emptyset$*.* $\mathcal{L}$ *is said to be the* viscous lattice of dilation δ.

Proof. *The class of sets invariants under* γ *is closed under union, as* γ *is an opening. On the other hand,* $\gamma(\cap X_i)$*,* $\{X_i,\ i \in I\} \in \mathcal{L}$ *is the largest element of* $\mathcal{L}$ *included in* $\cap X_i$*, therefore in each* X_i *which achieves the proof.* ■

Denote by $\mathcal{B}$=$\{\delta(x), x \in E\}$ the class of the dilates of the singletons. Relation (1) shows that class $\mathcal{B}$ is sup-generating for lattice $\mathcal{L}$; although the $\delta(x)$'s are not atoms in general. However, when E = $\mathbb{R}^n$ or $\mathbb{Z}^n$ and when dilation δ is invariant by translation, the elements of $\mathcal{B}$ are the translates of the image $B = \delta(O)$ of the origin. Then, if B is compact, then the associated viscous lattice is atomic.



a) b)

Figure 1. Counter example for the lack of distributivity in $\mathcal{L}$

Proposition 3 *The viscous lattice* $\mathcal{L}$ *of dilation* δ *is generally neither distributive nor co-primary, nor admits any unique complement.*

Proof. *Exhibiting a counter-example of each property is enough. Take for dilation* δ *the Minkowski's addition by the structuring element* B*, in* $\mathbb{R}^2$ *or in* $\mathbb{Z}^2$ *where* B *is the compact square of side 3. Consider the four squares* X, X', Y *and* Y' *depicted in figures 1a and 1b. These four elements are atoms* *of $\mathcal{L}$. However, a 3×3 square included in the union $(X \cup X')$ is not necessarily included in eitherX, or X' : therefore, atoms are not co-prime. Moreover,*

$$\begin{aligned} \text{in fig.1a } (X \wedge X') \cup Y &= \emptyset \cup Y \neq (X \cup Y) \wedge (X' \cup Y) = (X \cup Y) \\ \text{and in fig.1b } (X \cup X') \wedge Y' &= Y' \neq (X \wedge Y') \cup (X' \wedge Y') = \emptyset. \end{aligned}$$

hence lattice $\mathcal{L}$ is not distributive. Finally, the complements are multiple because when $X \in \mathcal{L}$, any set $Y \in \mathcal{L}$ such as

$$Y \supseteq X^c \qquad \text{and} \qquad \gamma\,(Y \cap X) = \emptyset.$$

is a complement of X in $\mathcal{L}$. ■

Such counter-properties are not independent: they illustrate *a contrario* G. Matheron's proposition ([1],p.101) according to which in a distributive lattice, any atom is co-prime. The main consequence of such a statement deals on operators, as it is not possible to base ourselves on the complement for defining pairs of dual operations. Fortunately, Galois' adjunctions between dilations and erosions, i.e. :

$$\delta\,(Y) \subseteq X \Longleftrightarrow \varepsilon\,(X) \subseteq Y \qquad\qquad X, Y \in \mathcal{L} \tag{3}$$

remain defined (as on any complete lattice) which will be useful.

2. Increasing Operations in $\mathcal{L}$

We will distinguish between the increasing operations in $\mathcal{L}$ and the "geodesic" type ones, where the marker, as well as the mask, may belong to $\mathcal{P}\,(E)$. This section is dedicated to the first category ; the second one is studied in section 4.

As all increasing operations in $\mathcal{L}$ are sup and inf of dilations and erosions of $\mathcal{L}$ into itself, we can just study these two operations only. Moreover, as $\mathcal{L}$ is a subset of $\mathcal{P}(E)$, we can wonder to which extent a dilation or an erosion on $\mathcal{P}(E)$ generate homologs on $\mathcal{L}$, and, if so, which ones? In order to avoid any confusion, we denote by α the current dilation of $\mathcal{P}(E)$ in itself, by β the adjoint erosion, and we keep symbol δ for referring to the fixed dilation of $\mathcal{P}(E)$ that generates $\mathcal{L}$.

Proposition 4 *1/ Any dilation $\alpha : \mathcal{P}(E) \to \mathcal{P}(E)$ that commutes with δ is also a dilation of $\mathcal{L}$ into itself.*

2/ If α^{-1} stands for the erosion adjoint to α in $\mathcal{P}(E)$, and β for the erosion adjoint to α in $\mathcal{L}$, then we have

$$\beta = \delta\varepsilon\alpha^{-1} \tag{4}$$

Proof. *1/ As dilations α and δ commute, we get*

$$\alpha\,[\mathcal{L}] = \alpha\,[\delta\,[\mathcal{P}(E)]] = \delta\,[\alpha\,[\mathcal{P}(E)]] \subseteq \mathcal{L}$$

and as the suprema in $\mathcal{L}$ *and in* $\mathcal{P}(E)$ *are the same,* α *is also a dilation of* $\mathcal{L}$ *in itself*

2/ The eroded $\beta(X)$*, in* $\mathcal{L}$*, is defined as the union of the sup-generators* $\delta(x)$ *so that* $\alpha[\delta(x)]$ *is included in* X*. Therefore*

$$\beta(X) = \cup\{\delta(x), \alpha[\delta(x)] \subseteq X\} = \cup\left\{\delta(x), \delta(x) \subseteq \alpha^{-1}(X)\right\},$$

which is nothing else than relation (4). ■

Therefore, each eroded element in $\mathcal{L}$ is equal to the opening by $\delta\varepsilon$ of the homologous eroded element (i.e.of the same adjoint dilation) in $\mathcal{P}(E)$.

3. Connections on Viscous Lattices

In Mathematical Morphology, a *connection* or *connected class* on $\mathcal{P}(E)$ is a set family $\mathcal{C} \subseteq \mathcal{P}(E)$ that satisfies the three following axioms (cf [4])

$$\begin{aligned} i/\ \emptyset &\in \mathcal{C} \\ ii/\ x &\in E \Longrightarrow \{x\} \in \mathcal{C} \quad \text{(class } \mathcal{C} \text{ is sup generating)} \\ iii/\ \{X_i, i \in I\} &\in \mathcal{C} \ \text{ and} \cap X_i \in \mathcal{C} \quad (\mathcal{C} \text{ is conditionally closed under union).} \end{aligned}$$

This definition extends to any complete sup-generated lattice (cf.[5]) by changing, in the first axiom, $\emptyset$ by the zero of the lattice, then, in the third one, the intersection and the union by the sup and the inf. The second axiom consists in stating that $\mathcal{C}$ is a sup-generating class.

In the present case, the analysis can be improved, for the class $\mathcal{B}$ of the structuring elements plays a specific role. Not only is it sup-generating, but it also belongs to all the sup-generating classes of $\mathcal{L}$ which are not too pathological. More particularly, in case of translation invariance, the $B \in \mathcal{B}$ become atoms, which ensures us that they belong to all the connected classes. Therefore, we can make more precise the general definition of a connection as follows

Definition 5 *Let* $\mathcal{L}$ *be a viscous lattice on* $\mathcal{P}(E)$ *of dilation* δ*. A class* $\mathcal{C}'$ *of* $\mathcal{L}$ *defines a connection on* $\mathcal{L}$ *when*

$$\begin{aligned} i/\ \emptyset &\in \mathcal{C}' \\ ii/\ x &\in E \Longrightarrow \delta\{x\} \in \mathcal{C}' \\ iii/\ \{X_i, i \in I\} &\in \mathcal{C}' \text{ and } \wedge X_i \neq \emptyset \Longrightarrow \cup X_i \in \mathcal{C}' \end{aligned} \qquad (5)$$

The correspondence between the two systems of axioms is so direct that we can wonder whether the restriction to $\mathcal{L}$ of any connection on $\mathcal{P}(E)$ does not induce a connection on $\mathcal{L}$ itself. That is the case, but it does not mean that the connections reached by this way are the most pertinent ones.

Proposition 6 *Let* $\mathcal{L}$ *be a viscous lattice on* $\mathcal{P}(E)$ *of dilation* δ*. A class* $\mathcal{C}' \subseteq \mathcal{L}$ *is a connection on* $\mathcal{L}$ *if it is the restriction to* $\mathcal{L}$ *of the union of a connection* $\mathcal{C}$ *on* $\mathcal{P}(E)$ *with the image* $\mathcal{B}$ *of the singletons of* $\mathcal{P}(E)$ *by* δ*, i.e. if*

$$\mathcal{C}' = (C \cap \mathcal{L}) \cup \mathcal{B} = (C \cup \mathcal{B}) \cap \mathcal{L} \qquad (6)$$

Proof. *Any class $\mathcal{C}'$ that satisfies rel.(6) satisfies also the first two axioms of definition 3. As regarding the third one, consider a family $\{X_i, i \in I\}$ in such a class $\mathcal{C}'$. When $\wedge X_i \neq \emptyset$, we have $\wedge X_i \neq \emptyset \Longrightarrow \cap X_i \neq \emptyset \Longrightarrow \cup X_i \in \mathcal{C}$, but $\cup X_i$ also belongs to $\mathcal{L}$, hence $\cup\ X_i \in \mathcal{C}'$, which is therefore a connection.* ■

Particularly, when connection $\mathcal{C}$ preserves the singletons, i.e.when $\delta\{x\}$ is $\mathcal{C}$-connected for any $x \in E$, then $\mathcal{C}'$ turns out to be the restriction of connection $\mathcal{C}$ to the lattice $\mathcal{L}$, and rel.((6)becomes

$$\mathcal{C}' = (\mathcal{C} \cap \mathcal{L}) \tag{7}$$

Conversely, let $\mathcal{C}'$ be a connection on $\mathcal{L}$. Associate it with the class $\mathcal{C}'_1$ conditionally closed under union generated by $\mathcal{C}'$, i.e. the class composed of all the unions of elements from $\mathcal{C}'$, whose intersection is not empty. We have $\mathcal{C}' \subseteq \mathcal{C}'_1$, for inf empty families may have a non empty intersection. Add to $\mathcal{C}'_1$ the class $\mathcal{S}$ of the singletons of $\mathcal{P}(E)$,by putting

$$\mathcal{C} = \mathcal{C}'_1 \cup \mathcal{S} \tag{8}$$

Class $\mathcal{C}$ is, by construction, a connection on $\mathcal{P}(E)$ (here we find one of the set connections mentioned by Ronse in [3]). More precisely, it is the smallest extension of $\mathcal{C}'$ to lattice of the connections on $\mathcal{P}(E)$, but obviously, it is not the only possible one: the maximum connection on $\mathcal{P}(E)$, that is to say $\mathcal{P}(E)$ itself, also includes $\mathcal{C}'$.
Note also that the restriction of $\mathcal{C} \cup \mathcal{B}$ on $\mathcal{L}$, where $\mathcal{C}$ is defined by rel (8), does not restore the connection $\mathcal{C}'$ where we started from to construct $\mathcal{C}$. In other words, proposition 6 does not draw up a complete inventory of the connections on $\mathcal{L}$, but rather describe those which do not really bring into play the inf of lattice $\mathcal{L}$.

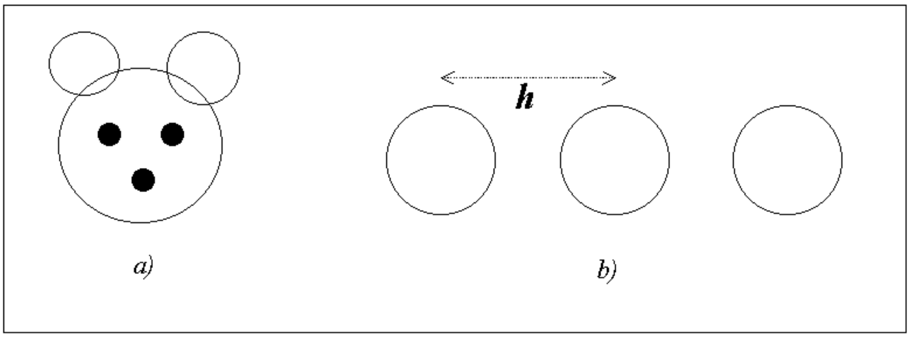


Figure 2. a) Am I connected? b) The two discs on the left hand side, or on the right hand side, form a connected set, but not the union of the three discs.

For instance, if we take for $\mathcal{C}$ the arc connection in $\mathbb{R}^2$ or in $\mathbb{Z}^2$, and for $\mathcal{L}$ the lattice of the sets open by the unit disk B, the union of the three lobes in figure

2a is an element of $\mathcal{C}$ and of $\mathcal{L}$ at the same time, in the sense of rel.(6). However, in the lattice $\mathcal{L}$ of the open sets, these three lobes are two by two disjoint. And they are disjoint because their erosion by B are disjoint in $\mathcal{P}(E)$. Therefore, figure 2a is considered as three separated particles for the connections on $\mathcal{L}$ which are based connectivity state in $\mathcal{P}(E)$ *before* the dilation δ. This can be carried out in the following way :

Theorem 7 *Let $\mathcal{C}$ be a connection on $\mathcal{P}(E)$ and $\delta : \mathcal{P}(E) \to \mathcal{P}(E)$ be a dilation, of adjoint erosion ε, that generates lattice $\mathcal{L}$. Then the image $\mathcal{C}' = \delta(\mathcal{C})$ of the connection $\mathcal{C}$ is a connection on the lattice $\mathcal{L}$.*

Proof. *As connection $\mathcal{C}$ contains the singletons and the empty set, and that $\delta(\emptyset) = \emptyset$, the first two axioms of definition 3 are confirmed. For proving the third one, we consider a family $\{X_i\}$ in $\mathcal{C}'$. For each X_i there is at least one Y_i such that $X_i = \delta\varepsilon(Y_i)$ with $\varepsilon(Y_i)$ connected in the sense of $\mathcal{C}$, as $\mathcal{C}'$ is the image of the connection $\mathcal{C}$ by δ. Now,*

$$\wedge X_i \neq \emptyset \Longleftrightarrow \delta\varepsilon(\cap X_i) \neq \emptyset \Longleftrightarrow \delta[\cap\varepsilon(X_i)] \neq \emptyset \Longrightarrow \cap\varepsilon(X_i) \neq \emptyset.$$

Therefore, the union $X = \cup X_i = \delta[\cup\varepsilon(Y_i)]$ belongs to class $\mathcal{C}'$ as the dilated from an element of $\mathcal{C}$. ■

Theorem 7 opens on new connections on $\mathcal{L}$, noticeably more restrictive than those of proposition 6. For example, the three lobes of figure 2a are now three separated connected components.

One may notice the general nature of theorem 7, which does not even assume that the images of $\mathcal{C}$ elements are $\mathcal{C}$-connected. For instance, if in $\mathbb{R}^2$ equipped with the arc-wise connection, one takes for δ the dilation by a doublet of points from h apart, the two discs on the left hand side of figure 2b form a connected set, but which is not anymore connected when the third disc is added. However, it is known [5] that if δ preserves the singletons connectivity, i.e. if $x \in E \Longrightarrow \delta\{x\} \in \mathcal{C}$, then δ preserves the connectivity of the whole class C, i.e. $\mathcal{C}' = \delta(\mathcal{C}) \subseteq \mathcal{C}$. Then the $\mathcal{C}'$-elements are connected in both lattices $\mathcal{L}$ and $\mathcal{P}(E)$.

4. Geodesic Operations

The geodesic problem we examine here deals with the extraction of connected components that hit a given marker, and that are included in a given mask. In order to solve this problem, two new hypotheses about the structure of the generic dilation δ are necessary. Firstly, when $\mathcal{L}$ is equipped with a connection $\mathcal{C}'$, the image by δ of any connected component has to still be connected, i.e. $\delta(\mathcal{C}') \subseteq \mathcal{C}'$. Then, δ is said to *preserve the connection* $\mathcal{C}'$. This condition enables to characterize the connected openings of $\mathcal{C}'$ as functions of those of $\mathcal{C}$. If, in addition, we want to extract these components by geodesic reconstruction, i.e. by means of successive dilations, δ not only has to be extensive, but also has to spread as far as necessary under iteration. Let us develop these two points.

4.1 CHARACTERIZATION OF THE ELEMENTARY CONNECTED OPENINGS

We know [5] that in any lattice $\mathcal{L}$ which is sup-generated by a family $\mathcal{B}$ and equipped with a connection $\mathcal{C}'$, a dilation δ preserves the connection $\mathcal{C}'$ if the images of the sup-generators are connected, i.e. if $\delta(\mathcal{B}) \subseteq \mathcal{C}'$ (NB : the proposition 8 of [5] supposes that the elements of $\mathcal{B}$ are co-prime: it is easy to see that this hypothesis is not included in the demonstration, which is a good point here). In the present situation, things are becoming more precise: the following theorem can be stated

Theorem 8 *Let $\mathcal{L}$ be a viscous lattice on $\mathcal{P}(E)$ of generating dilation δ, and let $\mathcal{B}$ be the image of the singletons of $\mathcal{P}(E)$ by δ. If $\mathcal{P}(E)$ is equipped with a connection $\mathcal{C}$ that δ preserves, then the connection $\mathcal{C}'$ induced on $\mathcal{L}$, as in theorem 5, is also preserved by δ. Furthermore, if γ_x and $\gamma'_{\delta(x)}$ denote the elementary connected openings on $\mathcal{P}(E)$ and $\mathcal{L}$ respectively, then*

$$\delta\varepsilon\gamma_x = \gamma_x\delta\varepsilon = \gamma'_{\delta(x)} \qquad x \in E \tag{9}$$

Proof. *Every element $B \in \mathcal{B}$ is the image of a singleton $\{x\}$ by δ, which implies its $\mathcal{C}'$-connectivity. Likewise, set $\delta(B)$, image of the $\mathcal{C}$-connected element $B = \delta(x)$, is $\mathcal{C}'$-connected in its turn. Then, proposition 8 of [5] implies that dilation $\delta : \mathcal{L} \to \mathcal{L}$ preserves the connection $\mathcal{C}'$. Now, let us prove relation (10). Let $\delta(x) \subseteq Z \in \mathcal{L}$. As δ preserves connection $\mathcal{C}$, we know (prop.11 in [5]) that the opening $\delta\varepsilon$ acts independently on the various $\mathcal{C}$-components of any $Z \in \mathcal{P}(E)$, which results in the first equality (10). Moreover,*

$$\delta(x) \subseteq Z \Longleftrightarrow x \in \varepsilon(Z) \Longrightarrow x \in \gamma_x[\varepsilon(Z)] = \varepsilon\gamma_x(Z),$$

$$\text{hence } \delta(x) \subseteq \delta[\varepsilon\gamma_x(Z)] \subseteq \gamma'_{\delta(x)}(Z) \tag{10}$$

since $\gamma'_{\delta(x)}(Z)$ is defined as the union of all the elements of $\mathcal{C}'$ that contain $\delta(x)$ and that are included in Z. On the other hand, according to theorem 7, we have $\mathcal{C}' \subseteq \mathcal{C}$, which implies that $\delta\varepsilon\gamma'_{\delta(x)}(Z) = \gamma'_{\delta(x)}(Z) \subseteq \gamma_x(Z)$ and that the second one of the inclusions (10) reverses. This results in the second equality in relation (9). ■

Consequently, any $\mathcal{C}'$-particle is opened by $\delta\varepsilon$ of the corresponding $\mathcal{C}$-particle. If we have a mean to compute $\gamma_x(Z)$, then relation (10) supplies a simple algorithm to derive $\gamma'_{\delta(x)}(Z)$.

4.2 GEODESIC RECONSTRUCTION

The previous step links the connected openings of $\mathcal{C}'$ to those of $\mathcal{C}$. Alternately, we can search for a more direct way, and see how a marker may flood $\gamma'_{\delta(x)}(Z)$. Consider an element $X \in \mathcal{L}$ whose connected components we are to characterize, when they hit a set A, called *marker*, and when they are included in set Z, or *mask*. As A and Z may belong either to $\mathcal{L}$ or to $\mathcal{P}(E)$, we have to extend the inf operation between two elements to the case when only one of the two belongs to $\mathcal{L}$.

Definition 9 *The mapping* $\mathcal{L} \otimes \mathcal{P}(E) \to \mathcal{L}$ *denoted by* $\wedge$ *and called* mixed infimum *is defined by the relation*

$$X \wedge Z = \delta\varepsilon (X \cap Z) \qquad\qquad X \in \mathcal{L}, \qquad Z \in \mathcal{P}(E)$$

The mixed infimum between X and Z is the union of the elements of $\mathcal{B}$ which are included in both X and Z. As the unions of $B \in \mathcal{B}$ included in Z or in $\delta\varepsilon(Z)$ are the same, the mixed infimum between X and Z is nothing but the infimum in $\mathcal{L}$ between X and $\delta\varepsilon(Z)$.

Proposition 10 *Let* δ *be an extensive generating dilation on* $\mathcal{P}(E)$*, which preserves the connection* $\mathcal{C} \subseteq \mathcal{P}(E)$*. Given a mask* $Z \in \mathcal{P}(E)$ *and a component* $\mathcal{C}$*-connected* $A \in \mathcal{P}(E)$*, with* $A \subseteq Z$*, the conditional dilated element*

$$A_1 = \delta(A) \wedge Z \tag{11}$$

is $\mathcal{C}'$*-connected and included in mask* Z*.*

Proof. *If* A *is empty, then* A_1 *is empty, hence* $\mathcal{C}'$*-connected. If* $\delta(A) \wedge Z$ *is empty,* $A_1 = A$ *is still* $\mathcal{C}'$*-connected. If not, then the element* $\delta(A) \wedge Z$ *is the union of* $B_i \in \mathcal{B}$ *that are included in* $\delta(A)$ *and in* Z*. Let* B *be one of them. Both* $\delta(A)$ *and* $\delta(B)$ *are* $\mathcal{C}'$*-connected, and by extensivity of* δ*, the infimum* $\delta(A) \wedge \delta(B)$*, which contains* B*, is not empty, hence* $\delta(A) \cup \delta(B)$ *is* $\mathcal{C}'$*-connected. Then, the family of* $\mathcal{C}'$*-connected elements of* $\mathcal{L}$ $\{\delta(A) \cup \delta(B_i); B_i \subseteq \delta(A) \wedge Z\}$ *admits a non-empty infimum (as it includes* $\delta(A)$*), and therefore, the union of its terms, that is to say* A_1*, is* $\mathcal{C}'$*-connected. Finally, as* $\delta(A) \wedge Z \subseteq \delta(A) \cap Z$*, the union* A_1 *is included in* Z*.* ■

In relation (11), the term $\delta(A) \wedge Z$ can be empty. Also, Z can be chosen in $\mathcal{P}(E)$, or via its opening $\delta\varepsilon(Z)$, in $\mathcal{L}$: this will not change the value of the dilated element A_1 in relation (11). Finally, if A is not an element of $\mathcal{L}$, but only of $\mathcal{P}(E)$, but if we can find an A' $\in \mathcal{C}'$ so that $A \subseteq A' \subseteq Z$, then the conditional dilate A_1 of the relation (11) is an element of $\mathcal{C}'$ such that $A \subseteq A_1 \subseteq Z$. In this sense, set A is said to mark the component $\mathcal{C}'$-connected A_1. But this circumstance is not the most common one, as the counter-example of fig. 4 hereafter shows it.

We still have to check that after successive dilations of the marker A, we end up and find the $\mathcal{C}'$-component of Z which contains A. Here, the extensivity of δ is not enough. For instance, take for E a square D of side a in $\mathbb{R}^2$, and for dilation the operation

$$\begin{aligned} \delta(x) &= \{x\} \qquad\qquad x \in D/y \\ \delta(y) &= B(y) \end{aligned}$$

where y is the central point of the square D and where $B(y)$ is a disk of diameter $< a$ centered in y. It is obvious that, whatever point $x \in D$ we start from, the successive iterations $\delta^{(n)}(x)$ will never flood the whole square D, but at the most the disk $B(y)$. More particularly, if $\mathcal{P}(E)$ is equipped with a connection which makes D connected , then there is at least one connected component of $\mathcal{P}(E)$, namely D itself, which will never be accessed by means of iterated dilations of a marker.Consequently, we must demand more than the simple extensivity of δ. We call *complete extensivity* this stronger property, and define it as follows

Definition 11 *Let E be a topological space, and δ an extensive dilation from $\mathcal{P}(E)$ into itself. Dilation δ is said to be* completely extensive *when, for any point $x \in E$ and for any compact $K \subseteq E$, there is an integer n such that the n^{th} iteration of $\delta(x)$ contains K.*

If ζ stands for the mapping of $\mathcal{L}$ in itself, defined by rel.(11), i.e. $\zeta(A) = A_1$, then we have

Proposition 12 *Let E be a topological space, δ be a completely extensive dilation on $\mathcal{P}(E)$ that generates the viscous lattice $\mathcal{L}$, and $\mathcal{C}$ be a connection on $\mathcal{P}(E)$ which induces the connection $\mathcal{C}'$ on $\mathcal{L}$. Every $\mathcal{C}'$-connected component of $\mathcal{L}$ which is included in a compact set Z_0 of $\mathcal{P}(E)$, and which is marked by $A \in \mathcal{C}(\mathcal{L})$ can be written as*

$$Z = \zeta^{(n)}(A)$$

for a suitable integer n.

Proof. *from $\zeta(A) = A_1 = \delta(A) \wedge Z = \delta(A) \cap \delta\varepsilon(Z)$, we can write*

$$A \cap \delta\varepsilon(Z) \subseteq \zeta(A) = \delta(A) \cap \delta\varepsilon(Z).$$

As δ is completely extensive, there exists a certain integer n and a certain point $x \in A$ such that: $\zeta^{(n)}(A) = \gamma_x \delta\varepsilon(Z)$, that is to say, according to theorem 8, $\zeta^{(n)}(A) = \gamma'_{(\delta x)}(A)$. ■

5. Applications

5.1 SET INTERPOLATION

Consider a binary contour, partly identified by a dotted line (ex : the black points in fig.3). In the lattice $\mathcal{L}$, the opening of the complement of this object, if not empty, turns out to be made of one or two $\mathcal{C}'$-particles, depending on the size of δ. Figure 3 depicts various reconstructions of A inside $Z \in \mathcal{P}(\mathbb{Z}^2)$, when δ is the Minkowski's addition by a disk D with of variable radius k. When parameter k is small, the "fluidity" of the successive dilations allows to go between the pins of Z^c ; when k increases, these pins stop the reconstruction process, but if k goes on increasing, then markers cannot be found anymore in $\mathcal{L}$, and the reconstructed set is empty. Before reaching situation 3a , one can find a minimum radius $k_{\max}$ for which the reconstruction is the largest possible, without touching the field borders (fig.3b). Besides, notice that the mapping Z→ $\gamma_{\max}(Z)$ is an opening of $\mathcal{L}$ into itself, and that the approach is a non parametric one, as $k_{\max}$ varies according to Z.

If now the same experiment is carried out again, but with the field border as a marker instead of A, we get a minimum dilation radius $k'_{\max}$ which does not flood the inner part. Although obviously $k'_{\max} = k_{\max}$, both openings

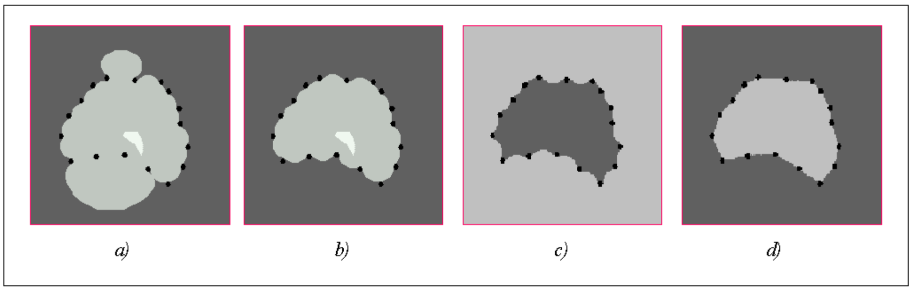

a) b) c) d)

Figure 3. a) In white, marker A, in black the complement of mask Z : geodesic dilations for a radius of viscosity k = 15 ; b) Case of the optimal radius $\mathrm{k}_{\max} = 17$; c) Optimal reconstruction from the edges of the field ; d) Corresponding median element.

are different (fig.3c); indeed they do *partition* the object Z, in the meaning of lattice $\mathcal{L}$.

For the sake of symmetry, we can interpolate between fig.3b and fig.3c by means of their median element (rel 13 in[6]):

$$M\left(X,Y\right) = \cup\left\{\left[X\cap Y \oplus k\mathcal{B}\right] \cap \left[\left(X\cup Y\right)\ominus k\mathcal{B}\right], k \geqslant O\right\} \qquad X,Y \in \mathcal{P}\left(E\right)$$

The interpolater M which we are led to, provides a nice interpolation from a visual point of view (fig.3d).

5.2 WATERSHED REGULARIZATION

Let us consider the positron image of a heart muscle, in figure 4a. The watershed line of its gradient enables the construction of a representative contour, but whose drawing is quite irregular.We purpose to regularize it by viscous flood. Let us consider the restriction of the gradient to the watershed line. This is a numerical function (looking a bit like the Great Wall of China) whose threshold at the value t gives the watershed points of gradient $\leq t$. Consequently, the successive thresholds appear like dotted lines with lower density as t increases. For each value t we determine a radius $k_{\max}$ of the dilation disc which enables the maximum reconstruction of the dotted line from the central marker. Finally we take the union of the maximum reconstruction as t varies: its represents the largest viscous surface built from the watershed (fig.4b).

As before, we are able to determine a maximum outer numerical function, from the borders of the field, then to take the support contours of both inner and outer surfaces (fig.4c) and to calculate their median element. The result, depicted in figure 4d, is less spectacular than the previous one, but realistic.

The approach of this second example is different from the one presented in [7], as the algorithm of [7] is not defined in the viscous lattices context. The choice for the markers is different too : in [7], they are purely internal, and the marker at level n is the reconstructed image of level $n-1$. Finally, the

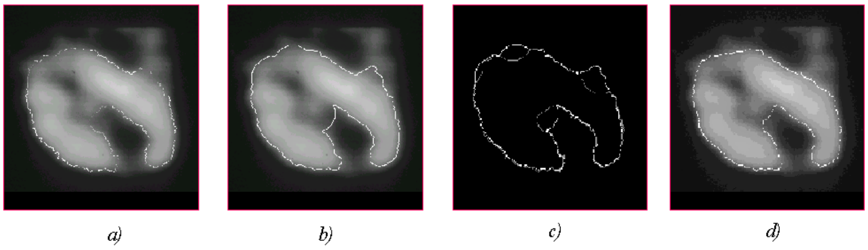


Figure 4. a) Positron image of a heart muscle and its contour by watershed ; b) Optimal reconstruction of the internal $\mathcal{C}'-$ components ; c) Internal and external optimal $\mathcal{C}'-$contours ; d) Median element.

third difference deals with the choice of the sizes of dilations δ: our algorithm is non parametric, therefore general, and could be particularized according to the type of the images investigated.

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