ALPHA-BETA FLAT ZONES, LEVELINGS AND FLATTENINGS

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Abstract Levelings are connected operators. On a digital grid a function g is an elementary leveling of f if and only if $g = (f \wedge \delta g) \vee \varepsilon g$ (where δg and εg are respectively the maximum and minimum of g on an elementary neighborhood of the grid). Such a leveling enlarges the flat zones : each flat zone of g is included in a flat zone of f. A more general leveling may be associated to each couple of an extensive dilation α and its adjunct anti-extensive erosion β. In the present paper, we show that such a leveling, defined by $g = (f \wedge \alpha g) \vee \beta g$ still is a connected operator. We describe the associated flat zones, and characterize floodings, razings, levelings and flattenings associated to α and β.

Keywords: Connection, flat zones, smooth zones, floodings, razings, levelings, flattenings.

1. Introduction

Connectivity plays an important role in mathematical morphology. Arc-wise connectivity is the most frequently used in practice, as it is at the base of many concepts: particles and holes for binary image, flat zones, regional minima or maxima.

A connected operator [6] enlarges the flat zones of f. The most widely used connected operators are reconstruction openings and closings. A function $g \leq f$ is a reconstruction opening (also called razing) of f if and only if $g = f \wedge \delta g$; similarly a function $g \geq f$ is a reconstruction closing of f (also called flooding) if and only if $g = f \vee \varepsilon g$ (where δ and ε are the elementary dilation and erosion on the grid). Recently levelings and flattenings have been introduced [4],[5],[3] and represent a symmetrical generalizations of reconstruction openings and closings. For instance g is an elementary leveling of f if and only if $g = (f \wedge \delta g) \vee \varepsilon g = (f \vee \varepsilon g) \wedge \delta g$. Formally a leveling may be associated to any extensive dilation α and its adjunct anti-extensive erosion β : g is a leveling of f if and only if $g = (f \wedge \alpha g) \vee \beta g$. In the present paper, we show that such a leveling still is a

H. Talbot, R. Beare (Eds): Proceedings of ISMM2002
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connected operator ; for this we have to characterize the flat zones which may be associated to α and β. This will be done in a first part. In a second part we define and characterize floodings, razings, levelings and flattenings associated to α and β. It appears that floodings, razings and levelings are indeed connected operators, whereas flattenings are only connected if the operators α and β are flat. In a last part, we show how to construct floodings, razings, flattenings and levelings of a function f associated to a marker function g. They appear as extremal elements in the lattice of all functions between f and g for the activity order.

2. Reminder on levelings and flattenings

2.1 Operators which never create transitions

Connected operators and in particular levelings are particularly useful in the framework of morphological segmentation. As a matter of fact, it is in general quite impossible to segment an image without beforehand simplifying it. Morphological segmentation works well if the image is composed of rather homogeneous zones separated by transition zones ; on the derived gradient image, the homogeneous zones appear as minima and the contours between them follow the watershed lines. Without an adequate filtering, most images are completely void of homogeneous regions, and segmentation will fail.

A good filter Φ should transform an image f into an image g with less details and simpler to segment. Furthermore, it would be nice to be able to discard f entirely and obtain a meaningful segmentation directly on g. This will be possible only if the contours of any segmentation produced on g exactly match identical contours in f. In other words, there should be no displacement of contours when one goes from f to g. Crossing a contour between two pixels p and q in the simplified image g means finding a kind of transition between two neighboring values g_p and g_q. A down transition between the neighboring pixels p and q will be written $g_p \succ g_q$. It may be any relation defined on the neighborhoods of g_p and g_q verifying $g_p \succ g_q \Rightarrow g_p > g_q$. In practice, a great variety of relations $\succ$ may be defined, each of them stressing different features. In this section we illustrate the basic notions with $g_p \succ g_q \Leftrightarrow g_p > g_q + \lambda$. As we require that no contour be displaced when going from f to g, to each transition between p and q for the image g should correspond a transition of f. Furthermore the amplitude of the transition between f_p and f_q should be larger than the amplitude of the transition between g_p and g_q. This leads to the definition of flattenings :

Definition 1 *A function g is a flattening of a function f if and only if for any couple of neighboring pixels (p, q):*

$$g_p - \lambda > g_q \Rightarrow g_p \neq g_q \Rightarrow \left[\begin{array}{c} f_p \geq g_p \text{ and} \\ g_q \geq f_q \end{array} \right] \text{ or } \left[\begin{array}{c} f_q \geq g_p \text{ and} \\ g_q \geq f_p \end{array} \right].$$

If furthermore, we ask that the direction of the transition remains the same we obtain the levelings :

Definition 2 *A function g is a leveling of a function f if and only if: for any couple of neighboring pixels (p, q): $g_p - \lambda > g_q \Rightarrow f_p \geq g_p$ and $g_q \geq f_q$*

The definition clearly shows that to any transition $g_p > g_q + \lambda$ corresponds an even bigger transition, since the interval $[g_q, g_p]$ is included in the interval $[f_q, f_p]$. The definition is illustrated in figure1, on two couples of pixels for which $f_s = g_s < g_t < f_t$ and $f_q < g_q < g_p = f_p$.

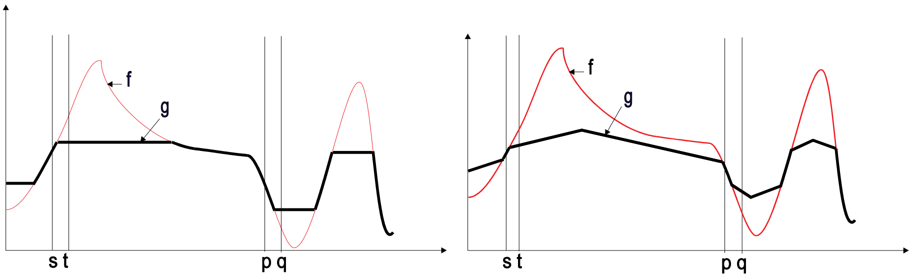

Figure 1. Illustration of the definitions of levelings: $g_p > g_q + \lambda \Rightarrow f_p \geq g_p$ and $g_q \geq f_q$ On the left $\lambda = 0$; On the right $\lambda = 1$

Let us analyze the relation defining levelings. The right part of the implication compares the functions f and g for the same pixels, whereas the left part compares the values taken by g on two distinct pixels, expressing the existence of a transition. In section 3 we will present a more general definition for transitions and derive from it general levelings and flattenings.

2.2 SMOOTH ZONES

The absence of down transition from g_p to g_q is expressed by $Not\,[g_p \succ g_q] \Leftrightarrow g_p \preccurlyeq g_q$. Two neighboring pixels without sharp transition between them are said to be at level:

$$\left| \begin{array}{c} g_p \preccurlyeq g_q \\ g_q \preccurlyeq g_p \end{array} \right| \Leftrightarrow g_p \Bumpeq g_q.$$

In the case where $g_p \succ g_q \Leftrightarrow g_p - \lambda > g_q$, then $g_p \Bumpeq g_q \Leftrightarrow |g_p - g_q| \leq \lambda$. Two types of uniform zones may then be defined : smooth zones and uniformly smooth zones. The relation $\Bumpeq$ is an adjacency relation between neighboring pixels (x, y), i.e. a reflexive and symmetrical relation : $x \Bumpeq x$ and $x \Bumpeq y \Leftrightarrow y \Bumpeq x$. But the relation $\Bumpeq$ is not transitive, i.e. it is not an equivalence relation. An equivalence relation $\bowtie$ is then derived : $x \bowtie y$ if and only if there exists a n-tuple of pixels $(p_1, p_2, ..., p_n)$ such that $p_1 = x$ and $p_n = y$, and for all i, (p_i, p_{i+1}) are neighbors and verify the symmetrical relation: $p_i \Bumpeq p_{i+1}$. We will call such a path $\Bumpeq$-connected path. The relation $\bowtie$ inherits the reflexivity and symmetry from the relation $\Bumpeq$. Transitivity is obtained through the concatenations of paths : if $x \bowtie y$ and $y \bowtie z$, then the concatenation of the $\Bumpeq$-connected paths between x and y on one hand and between y and z on the other hand produces a $\Bumpeq$-connected path between x and z.

Definition 3 *A set X will be a smooth zone if for any couple of pixels x and y in X, we have $x \bowtie y$. If X is a smooth zone and furthermore, we have $x \Bumpeq y$ for any couple of neighboring pixels x and y in X, then X is a uniformly smooth zone.*

The set of smooth zones associated to a function f form a connection [7]. In particular, if two smooth zones have a non empty intersection, their union also is a smooth zone.

In the case $\lambda = 0$,we obtain flat zones, which are both smooth and uniformly smooth. For $\lambda = 1$,

$$X = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

is a smooth zone since there exists a path with a slope smaller or equal to 1 between any couple of pixels. However, there exist within X sharp transitions between pixels, as between the values 1 and 4, hence X is not a uniformly smooth zone. This example also shows that uniformly smooth zones do not form a connection : the sets

$$X_1 = \begin{bmatrix} 3 & 2 \\ 4 & \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 3 & 2 \\ & 1 \end{bmatrix}$$

are each uniformly smooth, have a non empty intersection ; however, their union is smooth but not uniformly smooth.

Levelings and smooth zones. Suppose that g is a leveling of f. Then for a couple of neighboring pixels (p, q) $g_p \succ g_q$ implies $f_p \geq g_p$ and $g_q \geq f_q$, which in turn implies $f_p \succ f_q$. This means that $f_p \preccurlyeq f_q$ implies $g_p \preccurlyeq g_q$ and that $f_p \Bumpeq f_q$ implies $g_p \Bumpeq g_q$; hence if a set X is smooth (resp. uniformly smooth) for f it will also be smooth (resp. uniformly smooth) for g. This shows that levelings enlarge smooth and uniformly smooth zones.

Furthermore, they create new uniformly smooth zones, as we will see now. If for a couple of neighboring pixels (p, q) we have $g_q < f_q$, then the right-hand side of the implication $\{g_p \succ g_q \Rightarrow f_p \geq g_p \text{ and } g_q \geq f_q\}$ is false, implying that the left-hand side also is false : $g_p \preccurlyeq g_q$. Hence for any couple of neighboring pixels (p, q):

$$\left| \begin{array}{l} g_q < f_q \\ g_p < f_p \end{array} \right| \Rightarrow \left| \begin{array}{l} g_p \preccurlyeq g_q \\ g_q \preccurlyeq g_p \end{array} \right| \Rightarrow g_p \Bumpeq g_q.$$

In other words if g is a leveling of f, any connected particle X on which $g < f$ is uniformly smooth : there exists no transition between two neighboring pixels in X.

2.3 APPEARANCE OF AN EROSION AND A DILATION

Expressing $[g_p > g_q + \lambda \Rightarrow g_q \geq f_q]$ as $[\mathit{not}\,(g_p > g_q + \lambda) \text{ or } g_q \geq f_q]$ yields

$$[g_q \geq (g_p - \lambda) \text{ or } g_q \geq f_q]$$

that is $[g_q \geq f_q \wedge (g_p - \lambda)]$. Applying this criterion to all neighbors of the central pixel p yields: $[g \geq f \wedge (\delta g - \lambda)]$ which is equivalent to $[g \geq f \wedge (g \vee (\delta g - \lambda))]$.

The operator $\delta_\lambda = \mathrm{Id} \vee (\delta - \lambda)$ where Id represents identity is an extensive dilation, by a conic structuring function. In a dual way, we obtain $g \leq f \vee (g \wedge (\varepsilon g + \lambda))$, in which $\varepsilon_\lambda = \mathrm{Id} \wedge (\varepsilon + \lambda)$ is the anti-extensive erosion, dual of δ_λ.
Hence g will be a leveling of f if and only if $f \wedge \delta_\lambda g \leq g \leq f \vee \varepsilon_\lambda g$. But since $\varepsilon_\lambda g \leq g$ and $f \wedge \delta_\lambda g \leq g$, we obtain $(f \wedge \delta_\lambda g) \vee \varepsilon_\lambda g \leq g$. Symmetrically we obtain $g \leq (f \vee \varepsilon_\lambda g) \wedge \delta_\lambda g$. But since $(f \vee \varepsilon_\lambda g) \wedge \delta_\lambda g = (f \wedge \delta_\lambda g) \vee \varepsilon_\lambda g$, we finally get $g = (f \vee \varepsilon_\lambda g) \wedge \delta_\lambda g$. The inverse implications are easy to establish. Hence we obtain the criterion : g is a leveling of f if and only if $g = (f \vee \varepsilon_\lambda g) \wedge \delta_\lambda g$. This criterion allows to construct levelings associated to markers : if g is not a leveling of f, repeat the operation $g = (f \vee \varepsilon_\lambda g) \wedge \delta_\lambda g$ until stability.

Replacing (δ, ε) by an arbitrary couple (α, β) of an extensive dilation and its adjunct erosion still is a leveling $g = (f \vee \beta g) \wedge \alpha g$, first introduced and studied by G.Matheron in [3]. In the next sections we will construct this leveling as we did in the present section, deriving it from the definition of particular transitions.

2.4 Discussion

Levelings have been designed with the aim not to create sharp transitions between neighboring pixels on a digital grid. Crucial is the definition of a transition ; it relies on the choice of a neighborhood and of the relations between neighbors. This double choice appears in the couple of dilations and erosions which finally emerge in the characterization of levelings. In the example treated so far, we considered as neighbors the first neighbors of the grid and sharp transitions were defined as $(g_p \succ g_q \Leftrightarrow g_p > g_q + \lambda)$. This choice led to the dilation δ_λ and erosion ε_λ. Inversely, the dilation δ_λ conveys the information about both the neighborhood relations and on the definition of sharp transitions : let us consider the values of g at two pixels x and y. It is possible to establish whether there exists a sharp transition between g_x and g_y with the help of δ_λ. Let us consider the up-pulse function $\upharpoonleft_x^t$ defined by

$$\upharpoonleft_x^t(z) = \begin{cases} t & \text{if } z = x \\ -\infty & \text{if } z \neq x \end{cases}$$

The dilation $\delta_\lambda(\upharpoonleft_x^{g_x})$ produces a function equal to g_x at the pixel x, equal to $g_x - \lambda$ for all neighbors of x, and equal to $-\infty$ everywhere else. Hence g_x and g_y are comparable if and only if $\delta_\lambda(\upharpoonleft_x^{g_x})(y) > -\infty$ and $g_x \succ g_y$ if and only if $\delta_\lambda(\upharpoonleft_x^{g_x})(y) > g_y$ (see figure2A). In the following sections, we will use the same mechanism, in the framework of any completely ordered lattice, and not only on a digital grid, for defining sharp or smooth transitions, associated to any couple of an extensive dilation and its adjunct erosion. Then, we will define and study the properties of associated levelings and flattenings.

3. Up, down and smooth transitions in an image

3.1 REPRESENTATION OF DILATIONS AND EROSIONS WITH PULSE FUNCTIONS

We follow the presentation of Henk Heijmans in [1], pp.124–126.

Let $\mathcal{T}$ be some complete totally ordered lattice, and let $\mathcal{D}$,$\mathcal{E}$ be arbitrary sets. We call O the smallest element and Ω the largest element of $\mathcal{T}$. Fun($\mathcal{D}$,$\mathcal{T}$) represents the image defined on the support $\mathcal{D}$ with value in $\mathcal{T}$. We consider an arbitrary dilation α : Fun($\mathcal{D}$,$\mathcal{T}$) $\rightarrow$ Fun($\mathcal{E}$,$\mathcal{T}$) and its adjunct erosion β : Fun($\mathcal{E}$,$\mathcal{T}$) $\rightarrow$ Fun($\mathcal{D}$,$\mathcal{T}$)

For $h, x \in \mathcal{D}$ and $t \in \mathcal{T}$, the up-pulse function $\uparrow_h^t$ and the down-pulse function $\downarrow_h^t$ are defined by:

$\uparrow_h^t(x) = \left\{ \begin{array}{ll} t & \text{if } x = h \\ O & \text{if } x \neq h \end{array} \right\}$ and $\downarrow_h^t(x) = \left\{ \begin{array}{ll} t & \text{if } x = h \\ \Omega & \text{if } x \neq h \end{array} \right\}$

Every image f of Fun($\mathcal{D}$,$\mathcal{T}$) can be written $f = \bigvee\limits_{x \in \mathcal{D}} \uparrow_x^{f(x)} = \bigwedge\limits_{x \in \mathcal{D}} \downarrow_x^{f(x)}$. In other words up-pulse function comprise a sup-generating family and down-pulse functions an inf-generating family in the complete lattice Fun($\mathcal{D}$,$\mathcal{T}$). It is then possible to establish the following results :

Let $\mathcal{T}$ be some complete lattice, and let $\mathcal{D}$,$\mathcal{E}$ be arbitrary sets. The pair (β, α) forms an adjunction between Fun($\mathcal{E}$,$\mathcal{T}$) and Fun($\mathcal{D}$,$\mathcal{T}$) if and only if for every $x \in \mathcal{D}$, $y \in \mathcal{E}$ there exists an adjunction $(\beta_{y,x}, \alpha_{x,y})$ on $\mathcal{T}$ such that.

$\alpha\ f(y) = \bigvee\limits_{x \in \mathcal{D}} \alpha_{x,y}\,(f(x))$ and $\beta g(x) = \bigwedge\limits_{y \in \mathcal{E}} \beta_{y,x}\,(g(y))$

for $x \in \mathcal{D}$, $y \in \mathcal{E}$, $g \in$ Fun($\mathcal{E}$,$\mathcal{T}$) and $f \in$ Fun($\mathcal{D}$,$\mathcal{T}$).

Furthermore $\alpha_{x,y}\,(t) = \alpha(\uparrow_x^t)(y)$ and $\beta_{y,x}\,(s) = \beta(\downarrow_y^s)(x)$ for $t \in \mathcal{T}$

3.2 UP AND DOWN TRANSITIONS

Let f be a function of Fun($\mathcal{D}$,$\mathcal{T}$), α a dilation : Fun($\mathcal{D}$,$\mathcal{T}$) $\rightarrow$ Fun($\mathcal{E}$,$\mathcal{T}$) and its adjunct erosion β : Fun($\mathcal{E}$,$\mathcal{T}$) $\rightarrow$ Fun($\mathcal{D}$,$\mathcal{T}$), verifying for $O \leq k \leq \Omega$ $\{\alpha_{x,y}\,(k) \leq k \leq \beta_{x,y}(k)\}$ and $\{O < \alpha_{x,x}\,(k) = k = \beta_{x,x}(k) < \Omega\}$, which implies that α is extensive and β anti-extensive. We want to compare f_x and f_y for two arbitrary pixels x and y, with the help of α and β.

Dilating the pulse $\uparrow_x^{f(x)}$ with the dilation α produces a response different from O in some domain ; y should belong to this domain : $\{\alpha_{x,y}\,(f_x) > O\}$. Symmetrically, x should belong to the domain $\{\beta_{y,x}\,(f_y) < \Omega\}$.

Definition 4 *The values f_x and f_y taken by the function f at two pixels two pixels x and y are comparable if and only if $\alpha_{x,y}\,(f_x) > O$ and $\beta_{y,x}\,(f_y) < \Omega$.*

The relation “to be comparable” is not a symmetrical relation. It may happen that f_x and f_y are comparable and f_y and f_x are not comparable: we may have $\alpha_{x,y}\,(f_x) > O$ and $\alpha_{y,x}\,(f_y) = O$. The erosion β and dilation α do not need to be symmetrical for these definitions to be valid (α is symmetrical iff $\alpha_{x,y}\,(s) = \alpha_{y,x}\,(s)$). If f_x and f_y are comparable and simultaneously f_y and f_x also are comparable, we say that f_x and f_y are mutually comparable.

Since a and β form an adjunction, we have $\alpha_{x,y}\,(f_x) \leq f_y \Leftrightarrow f_x \leq \beta_{y,x}(f_y)$ or equivalently $\alpha_{x,y}\,(f_x) > f_y \Leftrightarrow f_x > \beta_{y,x}(f_y)$. This relation is at the basis of the relation <f_y is lower than f_x>:

Definition 5 *We say that f_y is lower than f_x and we write $f_y \sqsubset f_x$ if and only if $\{O \leq f_y < \alpha_{x,y}\,(f_x)$ and $\beta_{y,x}(f_y) < f_x \leq \Omega\}$ (1)*

In other words $f_y \sqsubset f_x$ if and only if f_x and f_y are comparable and $f_y < \alpha_{x,y}\,(f_x)$ (which is equivalent to $\beta_{y,x}(f_y) < f_x$). Notice that $f_y \sqsubset f_x$ is not a pre-order relation as it is not transitive : if f_x and f_y are comparable, f_y and f_z are comparable, then f_x and f_z are not necessarily comparable. Negating the relation $\{f_y \sqsubset f_x\}$ for comparable pixels yields :

Definition 6 *We say that f_y is greater or equal than f_x and we write $f_y \sqsupseteq f_x$ if and only if f_x and f_y are comparable and $f_y \geq \alpha_{x,y}\,(f_x)$ (which is equivalent to $f_x \leq \beta_{y,x}(f_y)$)*

Let f and g be two functions of Fun($\mathcal{D}$,$\mathcal{T}$) and (p, q) two pixels of T. It is easy to show that :

a) $\begin{array}{l} g_q \sqsubset g_p \\ f_q \sqsubset g_p \end{array} \Rightarrow (g_q \vee f_q) \sqsubset g_p$

b) $\begin{array}{l} g_p \sqsubset g_q \\ g_p \sqsubset f_q \end{array} \Rightarrow g_p \sqsubset (g_q \wedge f_q)$

c) $g_q \sqsubset g_p \leq f_p \Rightarrow g_q \sqsubset f_p$

Proof 7 *The proofs are easily obtained by choosing each time the adequate interpretation of $\sqsubset$. For instance let us prove (a).*
First of all, if g_q and g_p on one hand and f_q and g_p on the other hand are comparable, then $g_q \vee f_q$ and g_p also are comparable, since $g_q \vee f_q$ is equal either to g_q or f_q and $\alpha_{q,p}\,(g_q \vee f_q) > O$

$$\begin{array}{l} g_q \sqsubset g_p \\ f_q \sqsubset g_p \end{array} \Rightarrow \begin{array}{l} g_q < \alpha_{q,p}\,(g_p) \\ f_q < \alpha_{q,p}\,(g_p) \end{array} \Rightarrow (g_q \vee f_q) < \alpha_{q,p}\,(g_p) \Rightarrow (g_q \vee f_q) \sqsubset g_p$$

3.3 SIMILAR VALUES OF A FUNCTION AND FLAT ZONES

Let f be a function of Fun($\mathcal{D}$,$\mathcal{T}$). Combining the relations $\{f_y \sqsupseteq f_x\}$ and $\{f_x \sqsupseteq f_y\}$ yields a symmetrical relation, expressing that the values of f_y and f_x are similar:

Definition 8 *We define the similarity of f_y and f_x by :*
$\{f_x \Bumpeq f_y\} \Leftrightarrow \{O < \alpha_{x,y}\,(f_x) \leq f_y \leq \beta_{x,y}(f_x) < \Omega\} \Leftrightarrow$
$\{O < \alpha_{y,x}\,(f_y) \leq f_x \leq \beta_{y,x}(f_y) < \Omega\}$ *.We say that f_x and f_y are at level.*

From the hypothesis we made on α and β: $\{\alpha_{x,y}\,(k) \leq k \leq \beta_{x,y}(k)\}$ we conclude that if the function f takes the same value at two comparable pixels : $f_x = f_y = k$ and furthermore (f_x, f_y) and (f_y, f_x) comparable then $f_x \Bumpeq f_y$. The second hypothesis $\{O < \alpha_{x,x}\,(k) = k = \beta_{x,x}(k) < \Omega\}$ for $k \notin \{O, \Omega\}$ implies that $f_x \Bumpeq f_x$.

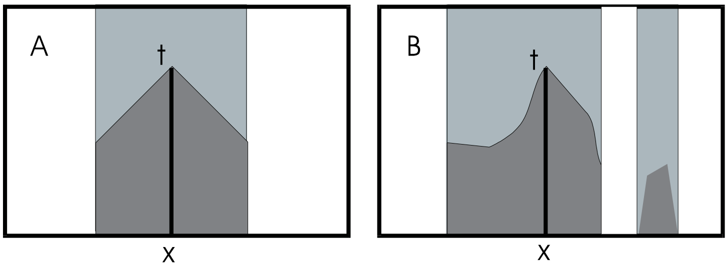

Figure 2. Two examples of a dilation of a pulse function $\upharpoonright_x^t$. A presents a symmetrical dilation by a cone. In B the structuring function is neither symmetrical nor connected. In the white zone are all pixels which are not comparable with x; in dark gray are all pixels below (x, t) and in light gray all pixels higher than or equal to (x, t).

Illustration. Figure2 and figure3 present examples of a dilation of a pulse function $\upharpoonright_x^t$. Figure2A presents a symmetrical dilation by a cone. In figure2B the structuring function is neither symmetrical nor connected. In the white zone are all pixels which are not comparable with x; in dark gray are all pixels below (x, t) and in light gray all pixels higher than or equal to (x, t) (including the frontier).

In figure3B we find the adjunct erosion of the down-pulse function $\downharpoonright_x^t$ in the white zone are all pixels which are not comparable with (x, t); in dark gray are all pixels above (x, t) and in light gray all pixels lower than or equal to (x, t). Figure3C presents in light gray the pixels which are at level with (x, t); it is the set of pixels which are neither above nor below (x, t); they are obtained by intersection of the light gray domains of figures 3A and 3B ; these two domains are symmetrical one from the other with respect to (x, t) ; for this reason, the domain of values at level with (x, t) is symmetrical with respect to (x, t).

We are now able to define smooth zones based on arc-wise connectivity.

Definition 9 *We say that two values f_x and f_y are smoothly linked and we write $f_x \bowtie f_y$ if there exists a series of pixels $\{x_0 = x, x_1, x_2, ...x_n = y\}$ such that $f_{x_i} \Bumpeq f_{x_{i+1}}$.*

Definition 10 *A set X is a smooth zone of an image f if and only if $f_x \bowtie f_y$ for any two pixels x and y in X.*

The relation $\bowtie$ is an equivalence relation. The associated equivalence classes are the maximal smooth zones. It is easy to verify that the smooth zones of f form a connection of $\mathcal{D}$.

Regional minima and maxima are easily defined :

Definition 11 *A set X is a regional minimum of an image f if and only if it is a smooth zone of f and for any two pixels x and y, $x \in X$ and $y \notin X$, such that f_x and f_y are comparable, we have $f_x \sqsubset f_y$.*

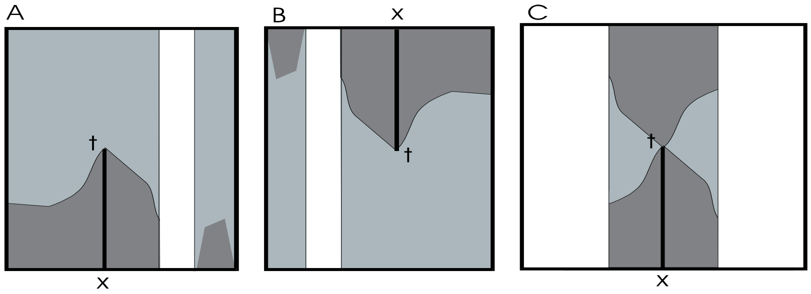

Figure 3. A: dilation of a pulse function $\uparrow_x^t$: in the white zone are all pixels which are not comparable with (x,t); in dark gray are all pixels below (x,t) and in light gray all pixels higher than or equal to (x,t).
B : adjunct erosion of the down-pulse function $\downarrow_x^t$: in the white zone are all pixels which are not comparable with (x,t) ; in dark gray are all pixels above (x,t) and in light gray all pixels lower or equal than (x,t).
C : in light gray the pixels which are at level with (x,t) ; in white are all pixels which are not mutually comparable with (x,t)

3.4 PARTICULAR CASES

We now check that our formulation is compatible with known classical situations, such as ordinary flat zones.

Classical digital flat zones. A binary set X may be characterized by its indicator function $1_X(y) = 1 \Leftrightarrow y \in X$. For the pair of elementary dilation and erosion (δ, ε), and two neighboring pixels x and y, we have $\delta_{y,x}(1_X(y)) = 1_X(y) = \varepsilon_{y,x}(1_X(y))$. Then $1_X(x) \asymp 1_X(y)$, expressed by $O < \delta_{y,x}(1_X(y)) \leq 1_X(x) \leq \varepsilon_{y,x}(1_X(y)) < \Omega$ implies $1_X(y) = 1_X(x)$, that is x and y both belong to X or both to $\overline{X}$. Similarly for functions $O < \delta_{y,x}(f_y) \leq f_x \leq \varepsilon_{y,x}(f_y) < \Omega$ means $O < f_y \leq f_x \leq f_y < \Omega$ that is $f_x = f_y$.

Flat zones associated to an arbitrary connection $\mathcal{C}$. Let us now consider an arbitrary connection $\mathcal{C}$ of a $\mathcal{D}$. We associate to it a dilation $\delta^{\mathcal{C}}$ defined as follows : $\delta^{\mathcal{C}}(x) = \bigcup \{C_i \in \mathcal{C} \mid x \in C_i\}$. Since the union of all C_i containing x is connected, $\delta^{\mathcal{C}}(x)$ is the largest connected set containing x. Let us write $\widehat{C_i}$ for the maximal connected sets. We then define: $\delta^{\mathcal{C}}(X) = \bigcup \left\{\delta^{\mathcal{C}}(x) \mid x \in X\right\} = \bigcup \{C_i \in \mathcal{C} \mid X \cap C_i \neq \emptyset\} = \bigcup \left\{\widehat{C_i} \in \mathcal{C} \mid X \cap \widehat{C_i} \neq \emptyset\right\}$.
Obviously $\delta^{\mathcal{C}}$ is increasing and commutes with union : it is indeed a dilation. $\delta^{\mathcal{C}}$ is also idempotent and extensive : it is also a closing. The fact that $\delta^{\mathcal{C}}$ also is a closing seems at first sight strange, as the class of invariants of a closing is stable by intersection. But the invariants of $\delta^{\mathcal{C}}$ are the maximal connected sets : and such sets are either identical or disjoint. Hence their class is stable by intersection.

Let us now study the erosion $\varepsilon^{\mathcal{C}}$ adjunct to $\delta^{\mathcal{C}}$: $Y \subset \varepsilon^{\mathcal{C}}(X) \Leftrightarrow \delta^{\mathcal{C}}(Y) \subset X$. Obviously $\varepsilon^{\mathcal{C}}(X) = \bigcup \left\{ Y \mid Y \subset \varepsilon^{\mathcal{C}}(X) \right\} = \bigcup \left\{ Y \mid \delta^{\mathcal{C}}(Y) \subset X \right\}$. But since $\delta^{\mathcal{C}}$ is extensive and idempotent:
$\bigcup \left\{ Y \mid \delta^{\mathcal{C}}(Y) \subset X \right\} = \bigcup \left\{ \delta^{\mathcal{C}}(Y) \mid \delta^{\mathcal{C}}(Y) \subset X \right\} = \bigcup \left\{ \widehat{C_i} \mid \widehat{C_i} \subset X \right\}$.

By duality $\varepsilon^{\mathcal{C}}$ is increasing, anti-extensive, idempotent and commutes with intersection : it is an erosion-opening.

Remark that $y \in \delta^{\mathcal{C}}(X) \Leftrightarrow \exists x \in X \mid y \in \delta^{\mathcal{C}}(x) \Leftrightarrow \exists x \in \delta^{\mathcal{C}}(y) \mid x \in X$. Hence we obtain the expression of the indicator function $1_{\delta^{\mathcal{C}}(X)} = \bigvee_{x \in \delta^{\mathcal{C}}(y)} 1_X(x)$. It is a function equal to the supremum of 1_X within each $\widehat{C_i}$. Similarly the indicator function $1_{\varepsilon^{\mathcal{C}}(X)}$ of $\varepsilon^{\mathcal{C}}(X)$ is a function equal to the infimum of 1_X within each $\widehat{C_i}$.

Repeating the same operations for all thresholds of a function f permits to define a dilation and an erosion of f associated to the connection $\mathcal{C}$: $\delta^{\mathcal{C}}(f)$ is equal to the supremum of f within each $\widehat{C_i}$ and the erosion $\varepsilon^{\mathcal{C}}(f)$ equal to the infimum of f within each $\widehat{C_i}$. It is then easy to verify that $\delta^{C}_{p,q}(f_p) = f_p$ if p and q belong to the same connected class and $\delta^{C}_{p,q}(f_p) = O$ if not. Similarly $\varepsilon^{C}_{p,q}(f_p) = f_p$ if p and q belong to the same connected class and $\varepsilon^{C}_{p,q}(f_p) = \Omega$ if not. For this reason, $f_p \Bumpeq f_q$ iff p and q belong to the same connected class of $\mathcal{C}$ and $f_p = f_q$

The lattice of partitions. The maximal connected classes of any connection form a partition : $x \in \delta^{\mathcal{C}}(x)$ and $\delta^{\mathcal{C}}(x) \cap \delta^{\mathcal{C}}(y) \neq \emptyset$ then $\delta^{\mathcal{C}}(x) = \delta^{\mathcal{C}}(y)$.

Let us now consider two distinct connections $\mathcal{C}_1$ and $\mathcal{C}_2$. An order relation between connections may be expressed as follows : $\mathcal{C}_1$ is coarser than $\mathcal{C}_2$ if for any point x we have $\delta^{\mathcal{C}_2}(x) \subseteq \delta^{\mathcal{C}_1}(x)$.

It is easy to see that $\delta^{\mathcal{C}_2} \wedge \delta^{\mathcal{C}_1}$ represents the dilation associated to the infimum of the partitions associated to $\mathcal{C}_1$ and $\mathcal{C}_2$. And the iteration until idempotence $\left(\delta^{\mathcal{C}_2}\delta^{\mathcal{C}_1}\right)^{\infty}$represents the dilation associated to the supremum of both partitions : it is the smallest partition D for which each class of $\mathcal{C}_1$ or $\mathcal{C}_2$ is contained in a class of D.

3.5 THE LATTICE OF SMOOTH ZONES

Let us consider two distinct couples of adjunct dilation and erosion : (α^1, β^1) and (α^2, β^2). Each of them induces on a function f a partition of chained flat zones, respectively the $\alpha^1\beta^1$-smooth-zones and $\alpha^2\beta^2$-smooth-zones.

The similarity relations between f-comparable pixels are respectively :

- $\left\{ f_x \Bumpeq^1 f_y \right\} \Leftrightarrow \left\{ O < \alpha^1_{x,y}(f_x) \leq f_y \leq \beta^1_{x,y}(f_x) < \Omega \right\}$
- $\left\{ f_x \Bumpeq^2 f_y \right\} \Leftrightarrow \left\{ O < \alpha^2_{x,y}(f_x) \leq f_y \leq \beta^2_{x,y}(f_x) < \Omega \right\}$

We say that the couple (α^1, β^1) is more active than the couple (α^2, β^2) if $O < \alpha^1_{x,y} \leq \alpha^2_{x,y} \leq \text{Identity} \leq \beta^2_{x,y} \leq \beta^1_{x,y} < \Omega$. In such a case $\left\{ f_x \Bumpeq^2 f_y \right\} \Rightarrow \left\{ f_x \Bumpeq^1 f_y \right\}$ implying that the smooth zones for (α^2, β^2) are included in the smooth zones for (α^1, β^1).

3.6 UNIFORMLY SMOOTH ZONES

As in section 2.2, we define uniformly smooth zones.

A set X is uniformly smooth if it is a smooth zone of an image f and if furthermore if $f_x \Bumpeq f_y$ for any two mutually comparable pixels x and y in X: $\{f_x \Bumpeq f_y\} \Leftrightarrow \{O < \alpha_{x,y}(f_x) \leq f_y \leq \beta_{x,y}(f_x) < \Omega\}$.

Let us suppose in the remaining part of this section that α and β are spatially symmetrical, i.e. they verify $O = \alpha_{x,y}(f_x) \Leftrightarrow \beta_{x,y}(f_x) = \Omega$, then any two comparable pixels are mutually comparable. Then for non comparable pixels we have $O = \alpha_{x,y}(f_x)$ and $\beta_{x,y}(f_x) = \Omega$ and $\alpha_{x,y}(f_x) \leq f_y \leq \beta_{x,y}(f_x)$ is also verified:

Definition 12 *A set X is a uniformly smooth zone of f if and only if for any couple pixels x and y in X we have $\{\alpha_{x,y}(f_x) \leq f_y \leq \beta_{x,y}(f_x)\}$.*

Hence if X is an $\alpha\beta$-smooth zone of f and $y \in X$, we have

$$\bigvee_{x \in X, x \neq y} \alpha_{x,y}(f_x) \leq f_y \leq \bigwedge_{x \in X, x \neq y} \beta_{x,y}(f_x).$$

But since $\alpha_{yy} f_y = \beta_{yy} f_y = f_y$, we also have

$$\bigvee_{x \in X} (f_x) \leq f_y \leq \bigwedge_{x \in X} \beta_{x,y}(f_x),$$

that is $\alpha_X(f_y) \leq f_y \leq \beta_X(f_y)$, where α_X and β_X are respectively the restrictions of α and β to X. But since α_X is extensive and β_X anti-extensive, we finally have $\alpha_X(f_y) = f_y = \beta_X(f_y)$. Inversely, it is obvious that if this last relation is true, then X is an $\alpha\beta$-smooth zone of f.

Remark 13 $\{\alpha_X(f_y) = f_y\} \Leftrightarrow \{f_y = \beta_X(f_y)\}$. *It is easy to establish : $\alpha_X(f_y) = f_y$ implies $\alpha_X(f_y) \leq f_y$, which by adjunction implies $f_y \leq \beta_X(f_y)$. But since β_X is anti-extensive, we obtain $f_y = \beta_X(f_y)$.*

Proposition 14 *A set X is a uniformly smooth zone of f if and only if $\{\alpha_X(f) = f\}$ or equivalently $\{f = \beta_X(f)\}$*

The counter-example of section 2.2 has shown that uniformly smooth zones of f do not form a connection : if X and Y are uniformly smooth zones of f, such that $X \wedge Y \neq \infty$, then $X \vee Y$ is not necessarily a uniformly smooth zone of f.

4. The levelings

4.1 UPPER LEVELINGS

Definition 15 *A function g is an upper-leveling of the function f if and only if for any pair of comparable pixels (p, q) the following implication holds:*
$g_q \sqsubset g_p \Rightarrow g_p \leq f_p$

The meaning of $g_q \sqsubset g_p$ being $\beta_{q,p} g_q < g_p$, the implication

$$g_p > \beta_{q,p}(g_q) \Rightarrow g_p \leq f_p$$

may be interpreted as $[g_p \leq \beta_{q,p}(g_q)$ *or* $g_p \leq f_p]$ that is $[g_p \leq f_p \vee \beta_{q,p}(g_q)]$. This criterion should be satisfied for any pair of comparable pixels (p,q). However since for non comparable pixels we have $\beta_{q,p}(g_q) = \Omega$, the criterion will be fulfilled for any couple of pixels:

Criterion up-lev2: *A function g is a upper-leveling of the function f if and only if for any pair of pixels (p,q) the following criterion holds:*
$g_p \leq f_p \vee \beta_{q,p}(g_q)$

Let us now fix the central pixel p and consider the set V_p of all pixels comparable with p. Repeating the criterion for all pixels x in V_p yields $g_p \leq f_p \vee \bigwedge_{x \in V_p} \beta_{x,p}(g_x)$. This inequality is equivalent to the following:

$$g_p \leq f_p \vee \left[g_p \wedge \bigwedge_{x \in V_p} \beta_{x,p}(g_x) \right]$$

(since $a < b \vee c$ is equivalent to $a < b \vee (c \wedge a)$). But $g_p = \beta_{p,p}(g_p)$. Hence we obtain another equivalent inequality, in which $T_p = p \cup V_p$:

$$g_p \leq f_p \vee \bigwedge_{x \in T_p} \beta_{x,p}(g_x) = g \leq f \vee \beta g.$$

The inverse is trivial. $g \leq f \vee \beta g$ implies

$$g_p \leq f_p \vee \bigwedge_{x \in V_p} \beta_{x,p}(g_x).$$

Hence we obtain another criterion for upper levelings:

Criterion up-lev3: *A function g is a upper-leveling of the function f if and only if* $g \leq f \vee \beta g$

Algebraic properties:

If g and h are both upper-levelings of the function f, then $f \vee h$, $f \wedge h$, $g \vee h$, $g \wedge h$ and the morphological center $[f \wedge (g \vee h)] \vee (g \wedge h)$ also are upper levelings of f.

Lets for instance prove that $g \vee h$ and $g \wedge h$ are upper-levelings : by criterion up-lev3 $g \leq f \vee \beta g$ and $h \leq f \vee \beta h$ which implies $g \vee h \leq f \vee \beta g \vee f \vee \beta h \leq f \vee \beta(g \vee h)$, since β is increasing. On the other hand $g \wedge h \leq (f \vee \beta g) \wedge (f \vee \beta h) = f \vee (\beta g \wedge \beta h) = f \vee \beta(g \wedge h)$, since β is an erosion, i.e commutes with the infimum.

4.2 FLOODINGS

The following characterization of floodings are all equivalent ; the first serving as definition :

A function g is an flooding of the function f if and only if

- *Flood1 :* $g \geq f$ *and* g *is an upper-leveling of the function* f
- *Flood2 :* $g \geq f$ *and for any pair of* comparable *pixels* (p, q) *:*
$\beta_{q,p} g_q < g_p \Rightarrow g_p = f_p$
- *Flood3 :* $g \geq f$ *and for any pair of* comparable *pixels* (p, q) *:*
$g_p \leq f_p \vee \beta_{q,p}(g_q)$
- *Flood4 :* $g = f \vee \beta g$

Algebraic properties:

If g and h are both floodings of the function f, then $g \vee h$, $g \wedge h$ also are floodings of f.

If h is an flooding of the function f and $f \leq g \leq h$, then h also is an flooding of the function g

4.3 Lower levelings and razings

Lower levelings and razings are the dual counterpart of respectively upper-levelings and floodings.We summarize here the definition and criteria.

Lower levelings:

A function g *is an lower-leveling of the function* f *if and only if :*

- *Low-lev1: for any pair of* comparable *pixels* (p, q) : $g_q < \alpha_{p,q} g_p \Rightarrow f_q \leq g_q$
- *Low-lev2: for any pair of* comparable *pixels* (p, q) : $f_p \wedge \alpha_{q,p}(g_q) \leq g_p$
- *Low-lev3:* $f \wedge \alpha g \leq g$

Razings:

A function g *is an razing of the function* f *if and only if:*

- *Raz1:* g *is an lower-leveling of the function* f *and* $g \leq f$
- *Raz2:* $g \geq f$ *and for any pair of* comparable *pixels* (p, q) *:*
$g_q < \alpha_{p,q} g_p \Rightarrow f_q = g_q$
- *Raz3: for any pair* (p, q) *of comparable pixels* $f_p \wedge \alpha_{q,p}(g_q) \leq g_p \leq f_p$
- *Raz4:* $g = f \wedge \alpha g$.

4.4 Levelings

Definition and criteria.

Definition 16 *A function* g *is an leveling of the function* f *if and only if* g *is both an upper and a lower leveling.*

Levelings are characterized by a number of criteria.(criteria Lev3, Lev6 and Lev7 are due to G. Matheron in [3])

A function g *is a leveling of the function* f *if and only if:*

- Lev1: for any pair of comparable pixels (p, q) :
$g_q \sqsubset g_p \Rightarrow f_q \leq g_q$ and $g_p \leq f_p$
- Lev2: *for any pair of pixels* (p, q) *:* $f_p \wedge \alpha_{q,p}(g_q) \leq g_p \leq f_p \vee \beta_{q,p}(g_q)$
- Lev3: $f \wedge \alpha g \leq g \leq f \vee \beta g$
- Lev4: $\left| \begin{array}{l} g_p > \beta g(p) \Rightarrow g_p \leq f_p \\ g_p < \alpha g(p) \Rightarrow f_p \leq g_p \end{array} \right|$
- Lev5: $\left| \begin{array}{l} f \vee g = f \vee \beta g \\ f \wedge g = f \wedge \alpha g \end{array} \right|$

- Lev6: $\left| \begin{array}{ll} \text{On } \{f \leq g\} & g = f \vee \beta g \\ \text{On } \{f \geq g\} & g = f \wedge \alpha g \end{array} \right|$
- Lev7: $g = (f \vee \beta g) \wedge \alpha g = (f \wedge \alpha g) \vee \beta g = \beta g \wedge_f \alpha g$

Remark : A function verifying criterion Flood 4 (resp. Raz4) obviously also verifies criterion Lev3 ; hence any flooding (resp. razing) also is a leveling.

Order relation between levelings.

Sequential levelings: The relation {to be a leveling of} is obviously reflexive ; hence it is a pre-order relation. Let us show that it is transitive, which means that it is a pre-order relation:.

Suppose that $\begin{array}{l} \{h \text{ leveling of } g\} \\ \{g \text{ leveling of } f\} \end{array}$; we want to prove that it implies $\{h$ leveling of $f\}$

Since h is a leveling of g, we have for any couple of comparable pixels x and $y : h_y \sqsubset h_x \Rightarrow g_y \leq h_y$ and $h_x \leq g_x$

But $h_y \sqsubset h_x$ means $h_y < \alpha_{x,y}(h_x)$. On the other hand, since $\alpha_{x,y}$ is increasing, $h_x \leq g_x$ implies $\alpha_{x,y}(h_x) \leq \alpha_{x,y}(g_x)$. Putting all inequalities together, we obtain $g_y \leq h_y < \alpha_{x,y}(h_x) \leq \alpha_{x,y}(g_x)$, yielding $g_y \sqsubset g_x$. But g being a leveling of f, $g_y \sqsubset g_x \Rightarrow f_y \leq g_y$ and $g_x \leq f_x$. Finally we get $f_y \leq g_y \leq h_y$. The other inequality $h_x \leq f_x$ is obtained by duality.

Parallel levelings: Suppose that we consider two levelings, the first based on the pair (α_1, β_1), the second on the pair (α_2, β_2) , verifying $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$. Then if g is a leveling of f for the pair (α_2, β_2) , then g also is a leveling of f for the pair (α_1, β_1) . This may be easily seen using criterion Lev3: $f \wedge \alpha_2 g \leq g \leq f \vee \beta_2 g \Rightarrow f \wedge \alpha_1 g \leq g \leq f \vee \beta_1 g$

Levelings and flat-zones.

Levelings are monotone planings: Suppose that g is a leveling of the function f. Let us show that $g_q \sqsubset g_p \Rightarrow f_q \sqsubset f_p$. But $g_q \sqsubset g_p$ means $g_q < \alpha_{p,q}(g_p)$and implies by definition of levelings $f_q \leq g_q$ and $g_p \leq f_p$. On the other hand, $\alpha_{p,q}$ being increasing, $g_p \leq f_p$ implies $\alpha_{p,q}(g_p) \leq \alpha_{p,q}(f_p)$
hence $f_q \leq g_q < \alpha_{p,q}(g_p) \leq \alpha_{p,q}(f_p)$, yielding $f_q \sqsubset f_p$. We have shown that the leveling is a monotone planing. In particular it is a planing, that is a connected operator : $f_x \Bumpeq f_y \Rightarrow g_x \Bumpeq g_y$

Levelings create flat-zones: The implication $g_q \sqsubset g_p \Rightarrow f_q \leq g_q$ and $g_p \leq f_p$ is equivalent to $f_q > g_q$ or $g_p > f_p \Rightarrow g_q \sqsupseteq g_p$
Now, if two comparable pixels (p, q) verify $f_q > g_q$ and $f_p > g_p$, we conclude that simultaneously $g_q \sqsupseteq g_p$ and $g_p \sqsupseteq g_q$, that is g_p and g_q are at level. That means that on $\{g > f\}$ (resp. $\{g < f\}$) any two mutually compatible pixels are at level. Any smooth zone of $\{g > f\}$ (resp. $\{g < f\}$) is uniformly smooth.

Levelings and regional minima. If (α, β) are flat operators, i.e. for comparable pixels p and q, we have $\alpha_{p,q}(t) = t = \beta_{p,q}(t)$, implying that $g_q \sqsupset g_p \Rightarrow g_q > g_p$. Then the leveling based on (α, β) does not create regional minima or maxima. More precisely if g is a leveling of f, and X a regional minimum of g, then there exists a set $Z \subset X$, which is a regional minimum for f. Let X be a regional minimum of g : for any couple of comparable pixels p and q, verifying $p \in X$ and $q \notin X$, we have $g_q > g_p$, which implies $f_q > f_p$, as g is a leveling of f. Now, consider the pixel x for which f is minimal within X. The flat zone Z containing x for the function f is necessarily contained in X, as the value of f increases on the outside boundary of X. By construction the value of f also increases on the outside boundary of Z, hence Z is a regional minimum of f. However, this is not true if (α, β) are not flat operators, as shows the following counter-example.

$$f = \begin{array}{ccccc} 9 & 9 & 9 & 9 & 9 \\ 9 & 3 & 1 & 2 & 9 \\ 9 & 2 & 4 & 3 & 9 \\ 9 & 9 & 9 & 9 & 9 \end{array} \text{ and } g = \begin{array}{ccccc} 9 & 9 & 9 & 9 & 9 \\ 9 & 4 & 4 & 4 & 9 \\ 9 & 4 & 4 & 4 & 9 \\ 9 & 9 & 9 & 9 & 9 \end{array}.$$

Obviously, g is a leveling of f for $(\alpha_\lambda f = f \vee (\delta f - \lambda), \beta_\lambda f = f \wedge (\delta f + \lambda)$. The pixels with value 4 form a regional minimum of g but do not contain a regional minimum for f.

5. The flattenings

A function g is a flattening of the function f if and only if any of the equivalent following criteria is verified

- Flat1: for any pair of comparable pixels (p,q) : $\begin{array}{c} (f \vee g)_q \sqsubset g_p \Rightarrow g_p \leq f_p \\ g_q \sqsubset (f \wedge g)_p \Rightarrow f_q \leq g_q \end{array}$
- Flat2: *for any pair of pixels (p,q) :* $f_p \wedge \alpha_{q,p}(f \wedge g)_q \leq g_p \leq f_p \vee \beta_{q,p}(f \vee g)_q$
- Flat3: $f \wedge \alpha(f \wedge g) \leq g \leq f \vee \beta(f \vee g)$
- Flat4: $\begin{array}{l} g_p > \beta(f \vee g)(p) \Rightarrow g_p \leq f_p \\ g_p < \alpha(f \wedge g)(p) \Rightarrow f_p \leq g_p \end{array}$
- Flat5: $\begin{array}{l} f \vee g = f \vee \beta(f \vee g) \\ f \wedge g = f \wedge \alpha(f \wedge g) \end{array}$
- Flat6: $\begin{array}{ll} \text{On } \{f \leq g\} & g = f \vee \beta(f \vee g) \\ \text{On } \{f \geq g\} & g = f \wedge \alpha(f \wedge g) \end{array}$
- Flat7: *for any pair of comparable pixels (p,q) :* $g_q \sqsubset g_p \Rightarrow \begin{array}{c} f_q \leq g_q \text{ and } g_p \leq f_p \\ \text{or} \\ f_q \sqsupseteq g_p \text{ and } g_q \sqsupseteq f_p \end{array}$

Remark: Comparing criterion Flat3 and Lev3 shows that any leveling is also an flattening, as

$$f \wedge \alpha(f \wedge g) \leq f \wedge \alpha g \leq g \leq f \vee \beta g \leq f \vee \beta(f \vee g).$$

5.1 ORDER RELATION BETWEEN FLATTENINGS

The relation {to be a flattening of} is obviously reflexive. It is easy to show that it is transitive if α and β are flat operators, in which case we have $g_q \sqsubset g_p \Rightarrow g_q < g_p$.

As shows the counterexample illustrated by figure4, a flattening not based on on flat operators α and β is not necessarily a planing. In this example h is a flattening of g since $h_q \sqsubset h_p \Rightarrow g_p \sqsubseteq h_q \sqsubset h_p \sqsubseteq g_q$. But from the right side of the expression one cannot conclude $g_p \sqsubset g_q$. On the contrary g_p and g_q are at level ; this shows that the flattening has created a transition in h, at a place where no transition is present for g.

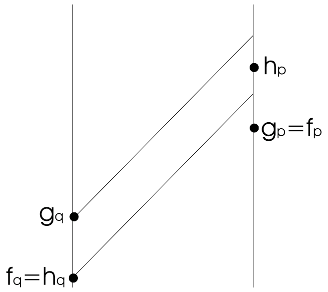

Figure 4. Example where $g_p \sqsubseteq h_q \sqsubset h_p \sqsubseteq g_q$ is verified but $g_p \sqsubset g_q$ is wrong.

Proposition 17 *If α and β are flat operators, then the relation {to be a flattening of} is transitive.*

5.2 FLATTENINGS AND FLAT ZONES

Are flattenings planings ?

Suppose that h is a flattening of g. h will be a planing if and only if for any couple of neighboring pixels (p, q) we have $g_p \Bumpeq g_q \Rightarrow h_p \Bumpeq h_q$.

This is equivalent to $h_q \sqsubset h_p \Rightarrow g_q \sqsubset g_p$ or $g_p \sqsubset g_q$.

Since h is a flattening of g, $h_q \sqsubset h_p \Rightarrow \begin{array}{l} g_q \leq h_q \sqsubset h_p \leq g_p \\ g_p \sqsubseteq h_q \sqsubset h_p \sqsubseteq g_q \end{array}$

As we have illustrated in the preceding section with figure4, $g_q \leq h_q \sqsubset h_p \leq g_p \Rightarrow g_q \sqsubset g_p$, however if α and β are not flat, it is not true that $g_p \sqsubseteq h_q \sqsubset h_p \sqsubseteq g_q$ implies $g_p \sqsubset g_q$. And indeed in the example of figure4, h is a flattening of g, (g_p, g_q) is a flat zone but (h_p, h_q) is not a flat zone. On the contrary, if α and β are flat operators, then $\sqsubset$ implies $<$ and $g_q \leq h_q < h_p \leq g_p \Rightarrow g_q < g_p$, showing that in this case flattenings are planings.

Proposition 18 *If α and β are flat operators, then flattenings are connected operators.*

Flattenings create flat-zones

Let g be an flattening of f. It is easy to verify that the zones where $\{g > f\}$ (resp. $\{g < f\}$) are flat. Indeed on $\{g > f\}$ we have $g = f \vee g = f \vee \beta(f \vee g)$, that is $f \vee g$ is a flooding of f. And a flooding, as any other leveling creates flat zones on its anti-extensivity zones.

6. The *f*-activity lattice

This section sets the framework in which we will work and introduces the f-activity lattice. It closely follows [3]

Definition 19 *For* $g, h, f \in T^E$, *we say* h *separates* g *and* f, *and we write* $(g\ \ h\ \ f)$ *or equivalently* $(f\ \ h\ \ g)$ *if and only if for any* $x \in E$, *the series* $(g_x\ \ h_x\ \ f_x)$ *is monotonous:* $\forall x \in E : g_x \leq h_x \leq f_x$ *or* $g_x \geq h_x \geq f_x$

Definition 20 *We will call* Inter (g, f) *the class of functions* $h \in T^E$, *which separate* g *and* f.

It is easy to prove that: $h \in \text{Inter}\,(g, f) \Leftrightarrow g \wedge f \leq h \leq g \vee f$

In particular, $g \wedge f$ is the smallest element of Inter (g, f) and $g \vee f$ its largest element. Inter (g, f) is obviously a quasi-sub lattice of T^E (i.e. it is a complete sub-lattice for the order induced by $\leq$ and the sup and inf in Inter (g, f) are identical to $\vee$ and $\wedge$ for the non empty sets.

6.1 The order $g >_f h$

Definition 21 *We say that* g *is more far away from* f *than* h, *or that* g *is bigger than* h in the order f *and we write* $g >_f h$ *if and only if* h *separates* g *and* f : $\qquad g >_f h \Leftrightarrow (g\ \ h\ \ f)$

Proposition 22 $>_f$ *is an order relation on* T^E, *and moreover*
$g >_f h \Leftrightarrow f >_g h$

Proposition 23 *With the order relation* $>_f$, T^E *becomes a complete* $\wedge-$*half lattice (but not a lattice). The infimum for the order relation* f *is written* $\wedge_f$ *and is given by*

$$\bigwedge_f h_i = \left(f \vee \left(\bigwedge h_i\right)\right) \wedge \left(\bigvee h_i\right) = \left(f \wedge \left(\bigvee h_i\right)\right) \vee \left(\bigwedge h_i\right).$$

Proposition 24 *For* $a, f \subset T^E$, Inter (a, f) *is a complete lattice for the order* f. *The function* a *is then the highest element. For any family* h_i *of* Inter (a, f)*:*

$$\bigwedge_f h_i = \begin{cases} \vee h_i & \text{on } \{a \leq f\} \\ \wedge h_i & \text{on } \{a \geq f\} \end{cases}$$

and

$$\bigvee_f h_i = \begin{cases} \wedge h_i & \text{on } \{a \leq f\} \\ \vee h_i & \text{on } \{a \geq f\} \end{cases}$$

On Inter (a, f) *it is also true that* $\bigvee_f h_i = \bigwedge_a h_i$

Corollary 25 *The relation* $\wedge_f$ *is associative.*

7. Minkowski subtractions

7.1 Levelings are Minkowski subtractions

Given a function f and a function g, let us introduce two types of translation which express the way the value of g at pixel q is perceived at position p. We define them as follows:

$$\overleftarrow{pq}(g;f) = \begin{cases} \beta_{q,p}(g_q) & \text{on } \{g_p > f_p\} \\ g_p & \text{on } \{g_p = f_p\} \\ \alpha_{q,p}(g_q) & \text{on } \{g_p < f_p\} \end{cases}$$

and

$$\underleftarrow{pq}(g;f) = \begin{cases} f_q \vee \beta_{q,p}(g_q) & \text{on } \{g_p > f_p\} \\ g_p & \text{on } \{g_p = f_p\} \\ f_q \wedge \alpha_{q,p}(g_q) & \text{on } \{g_p < f_p\} \end{cases}$$

This last translation has been by R.Kresh in [2] in the simple case of (δ, ε).
It is easy to verify that g is an leveling of the function f if and only if $g_p = \bigwedge_f \overleftarrow{pq}(g;f)$ and g is an flattening of the function f if and only if $g_p = \bigwedge_f \underleftarrow{pq}(g;f)$.
We remark that $\underleftarrow{pq}(g;f)$ is increasing for the order $<_f$: if $g^1 <_f g^2$ then $\underleftarrow{pq}(g^1;f) <_f \underleftarrow{pq}(g^2;f)$. Furthermore $\underleftarrow{pq}(g;f)$ commutes with $\wedge_f$. For this reason flattenings are erosions for the order $<_f$.

8. Construction of floodings, razings, levelings and flattenings

In sections 5 and 6, we have given a number of criteria, expressing that a function g is a flooding, razing, leveling or flattening of a function f. In this last section, we consider a pair of functions f and h and will study the family of floodings, razings, levelings and flattenings of f within $\operatorname{Inter}(f,h)$. The function h will be called marker and the function f reference function ; in the particular case of binary razings, we will find the reconstruction opening of a set from a marker. Floodings and razings are dual operators, so we will treat only floodings in detail ; the results for razings is obtained by duality. We will then treat the case of flattenings and end with the levelings.

8.1 CONSTRUCTION OF FLOODINGS AND RAZINGS

The family of floodings and razings of f contained in Inter (f,h).

The function g is an flooding of f iff $f \leq g \leq f \vee \beta g$. Now, let us study the floodings of f included in $\operatorname{Inter}(f,h)$. They verify $f \leq g \leq f \vee \beta g \leq f \vee h$. On the other hand, floodings of f form a lattice : if (g_i) is a family of floodings of f, then $\bigvee g_i$ also is a flooding of f. Hence the supremum (for both order relations $>$ and $>_f$) of the family of floodings of f belonging to $\operatorname{Inter}(f,h)$ is the largest flooding in $\operatorname{Inter}(f,h)$ and we will write $\operatorname{Fl}(f,h)$.

Now, how can we construct it ? If we write $h_0 = f \vee h$, we know that each flooding g of f verifies $f \leq g \leq f \vee \beta g \leq h_0$. But β being increasing, $g \leq h_0$ implies $\beta g \leq \beta h_0$; hence g also verifies $f \leq g \leq f \vee \beta g \leq f \vee \beta h_0$. Defining the recurrence $h_n = f \vee \beta h_{n-1}$, the same arguments produce $f \leq g \leq h_n$. But then $f \leq g \leq \bigwedge h_n$. β being an anti-extensive operator, the sequence h_n is decreasing and bounded by f. Its limit is equal to $h_\infty = \bigwedge h_n$; and this limit, verifying $f \leq h_\infty \leq f \vee \beta h_\infty$ is itself a flooding of f and is equal to $\operatorname{Fl}(f,h)$, the largest flooding of f within Inter (f,h) . For finite digital images, the limit is obtained by finite iteration until stability of $h_n = f \vee \beta h_{n-1}$, with $h_0 = f \vee h$. We recognize the usual reconstruction closing if $\beta = \varepsilon$.

Similarly the largest razing of f for the order relation $>_f$ in Inter (f,h), which is also the smallest razing for the order relation $>$ is equal to $\bigvee h_n$, where $h_n = f \wedge \alpha h_{n-1}$, with $h_0 = f \wedge h$; we write $\mathrm{Rz}(f,h)$. In the case of finite digital images, this supremum is obtained by finite iteration until $h_{n+1} = h_n$.

Properties of $\mathrm{Rz}(f,h)$ and $\mathrm{Fl}(f,h)$. For the order relation $>_f$ and considered as a function of h, $\mathrm{Rz}(f,h)$ and $\mathrm{Fl}(f,h)$ are increasing, anti-extensive and idempotent: they are openings.

8.2 CONSTRUCTION OF FLATTENINGS

The family of flattenings of f contained in Inter (f,h). Let us now study the family of flattenings of f contained in Inter (f,h). A function g is an flattening of the function f if it verifies criterion Flat5:

$$\left| \begin{array}{l} f \vee g = f \vee \beta(f \vee g) \\ f \wedge g = f \wedge \alpha(f \wedge g) \end{array} \right|$$

which expresses that $f \vee g$ is a flooding of f and $f \wedge g$ a razing of f. Hence g will be the largest flattening of f for $<_f$ in Inter (f,h) if and only if $f \vee g$ is equal to $\mathrm{Fl}(f,h)$ the largest flooding of f in Inter (f,h) and $f \wedge g$ equal to $\mathrm{Rz}(f,h)$, the largest razing of f for the order relation $>_f$ in Inter (f,h). Hence the largest flattening of f in Inter (f,h), which will be written $\Xi(f,h)$ or $\Xi_h f$ is equal to $\mathrm{Rz}(f,h)$ on $\{h \leq f\}$ and $\mathrm{Fl}(f,h)$ on $\{h \geq f\}$.

Properties of $\Xi(f,h)$. For the order relation $>_f$ and considered as a function of h, $\Xi(f,h)$ is increasing, anti-extensive and idempotent: it is an opening.

8.3 CONSTRUCTION OF LEVELINGS

The leveling of f associated to h.. There is no extremal leveling for the order $>_f$ in Inter (f,h) . As a matter of fact, the supremum $\vee_f$ of two levelings is not necessarily a leveling but a flattening, since levelings are particular flattenings and the supremum $\vee_f$ of two flattenings is a flattening. In figure5 g and h are both levelings of f, but $g \vee_f h$ is not a leveling of f but only a flattening.

However, the following lemma shows that if h is of the form $\alpha k \wedge_f \beta k$, then all flattenings contained in Inter (f,h) are levelings.

Lemma 26 *If a function g is a flattening of f and verifies $f \wedge \alpha k \leq g \leq f \vee \beta k$, i.e. $g \in$ Inter $(f, \alpha k \wedge_f \beta k)$ for a given function k, then g is a leveling of f.*

Hence, the largest flattening for $>_f$ contained in Inter $(f, \alpha h \wedge_f \beta h)$ is in fact a leveling of f ; it is equal to $\Xi(f, \alpha h \wedge_f \beta h)$. We will define it as the leveling of f associated to the marker h and write $\Lambda(f,h)$.

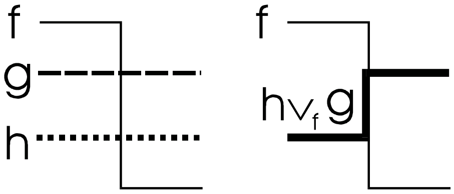

Figure 5. g and h are both levelings of f, but $g \vee_f h$ is not a leveling of f but only a flattening

Construction of $\boldsymbol{\Lambda(f,h)}$. The construction of $\Lambda(f,h)$ is derived from its definition $\Xi(f, \alpha h \wedge_f \beta h)$. Defining

$$g = \alpha h \wedge_f \beta h = \left| \begin{array}{c} f \vee \beta h \text{ on } \{h \geq f\} \\ f \wedge \alpha h \text{ on } \{h \leq f\} \end{array} \right|,$$

we have to construct the $_{>f}$−largest flattening of f in Inter (f,g). As we have seen in the previous section, the first step for constructing the flattening is

$$\left| \begin{array}{c} \text{on } \{g \geq f\} = \{h \geq f\} : g^1 = f \vee \beta\,(f \vee g) \\ \text{on } \{g \leq f\} = \{h \leq f\} : g^1 = f \wedge \alpha\,(f \wedge g) \end{array} \right|.$$

As a matter of fact, this first step and the next steps may be simplified into

$$\left| \begin{array}{c} \text{on } \{g \geq f\} = \{h \geq f\} : g^n = f \vee \beta g^{n-1} \\ \text{on } \{g \leq f\} = \{h \leq f\} : g^n = f \wedge \alpha g^{n-1} \end{array} \right|$$

for $n \geq 1$.

If we write $g_+^0 = f \vee \beta h$, we define $\Lambda_h^+ f = \bigwedge g_+^n$, where g_+^n is defined by the recurrence $g_+^n = f \vee \beta g_+^{n-1}$ and $\Lambda_h^- f = \bigvee g_-^n$, where $g_-^n$is defined by the recurrence $g_-^n = f \wedge \alpha g_-^{n-1}$, $g_-^0 = f \wedge \alpha h$. Then the levelings are obtained by

$$\left| \begin{array}{c} \text{on } \{h \geq f\} : \Lambda_h^+ f \\ \text{on } \{h \leq f\} : \Lambda_h^- f \end{array} \right|.$$

For a fixed h, $\Lambda_h^- f$, as a function of f, is increasing, anti-extensive and idempotent : it is an opening of f. For a fixed h, $\Lambda_h^+ f$, as a function of f, is increasing, extensive and idempotent : it is a closing of f.

It follows that

$$\Lambda_h^+ f = \left| \begin{array}{c} \Lambda_h f \text{ on } \{h \geq f\} \\ f \text{ on } \{h \leq f\} \end{array} \right|$$

and

$$\Lambda_h^- f = \left| \begin{array}{c} f \text{ on } \{h \geq f\} \\ \Lambda_h f \text{ on } \{h \leq f\} \end{array} \right|.$$

If now, for a fixed h, we construct $\Lambda_h^- \Lambda_h^+ f$, and apply the preceding relations, we find

$$\Lambda_h^- \Lambda_h^+ f = \left| \begin{array}{c} \Lambda_h^+ f \text{ on } \left\{h \geq \Lambda_h^+ f\right\} \\ \Lambda_h \Lambda_h^+ f \text{ on } \left\{h < \Lambda_h^+ f\right\} \end{array} \right|.$$

But on $\left\{h \geq \Lambda_h^+ f\right\}$, we have on $h \geq \Lambda_h^+ f \geq f$; hence $\left\{h \geq \Lambda_h^+ f\right\} \subset \{h \geq f\}$,where $\Lambda_h^+ f = \Lambda_h f$

On the other hand, on $\left\{h < \Lambda_h^+ f\right\}$, we have $h < \Lambda_h^+ f \leq f \vee h$ which implies $\Lambda_h^+ f \leq f$. But $\Lambda_h^+ f \geq f$, implying $\Lambda_h^+ f = f$ on $\left\{h < \Lambda_h^+ f\right\}$. and $\Lambda_h \Lambda_h^+ f = \Lambda_h f$ on $\left\{h < \Lambda_h^+ f\right\}$. Finally we have $\Lambda_h^- \Lambda_h^+ f = \Lambda_h f$ everywhere.

In a similar way we may prove that $\Lambda_h^+ \Lambda_h^- f = \Lambda_h f$.This shows that the leveling $\Lambda_h f$, as a function of f is a commutative product of an opening by a closing, that is a strong filter.

Illustration We compare two levelings on the same reference image f (see figure6_left) and marker image h (not illustrated here : it is completely black with a white dot on the left hand of the girl). The first leveling $\Lambda_h f$ is associated to the dilation δ^{++} and its adjunct erosion ε^{--} and is illustrated in figure6_center ; the dilation δ^{++} is the supremum of the dilation δ with a dilation of a couple of pixels, the first 5 pixel on the left side of the origin and the second 5 pixels on the right side of the origin. The δ part cares for the normal connectivity reconstructions whereas the couple of added pixels permits jumps from one zone to another. Indeed the ordinary leveling shown in figure6_right is less satisfactory as it is unable to reconstruct some parts of the image, although it uses the same marker ; it is unable to jump from on book to the next on the shelve in the background.

Figure 6. Left : f =original image. The marker image h is completely black with a white dot on the left hand of the girl.
Center : leveling $\Lambda_h f$ associated with the dilation δ^{++} and erosion ε^{--};
Left : leveling $\Lambda_h f$ associated with the dilation δ and erosion ε ; without jumps, the reconstruction is much less complete (see for instance the books)

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