Modelling Operational Risk using Bayesian Approach

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- **Basel II recognises the importance of the potential impact of losses due to Operational Risk and requires that banks hold adequate capital to protect against these losses.**

- **In Australia, the national regulator (APRA) is now applying the same detailed scrutiny to Operational Risk as previously to credit risk and market risk.**

- **The BCBS (Basel Committee on Banking Supervision) defined Operational Risk as: *the risk of loss resulting from inadequate or failed internal processes, people and systems, or from external events.* Includes legal, excludes strategic and reputational risks**

Under the Basel II framework, banks can estimate operational risk using one of three approaches:

- **The Basic Indicator Approach**
  (15% of Gross income averaged over three year)

- **The Standardised Approach**
  (12-18% on the business line level)

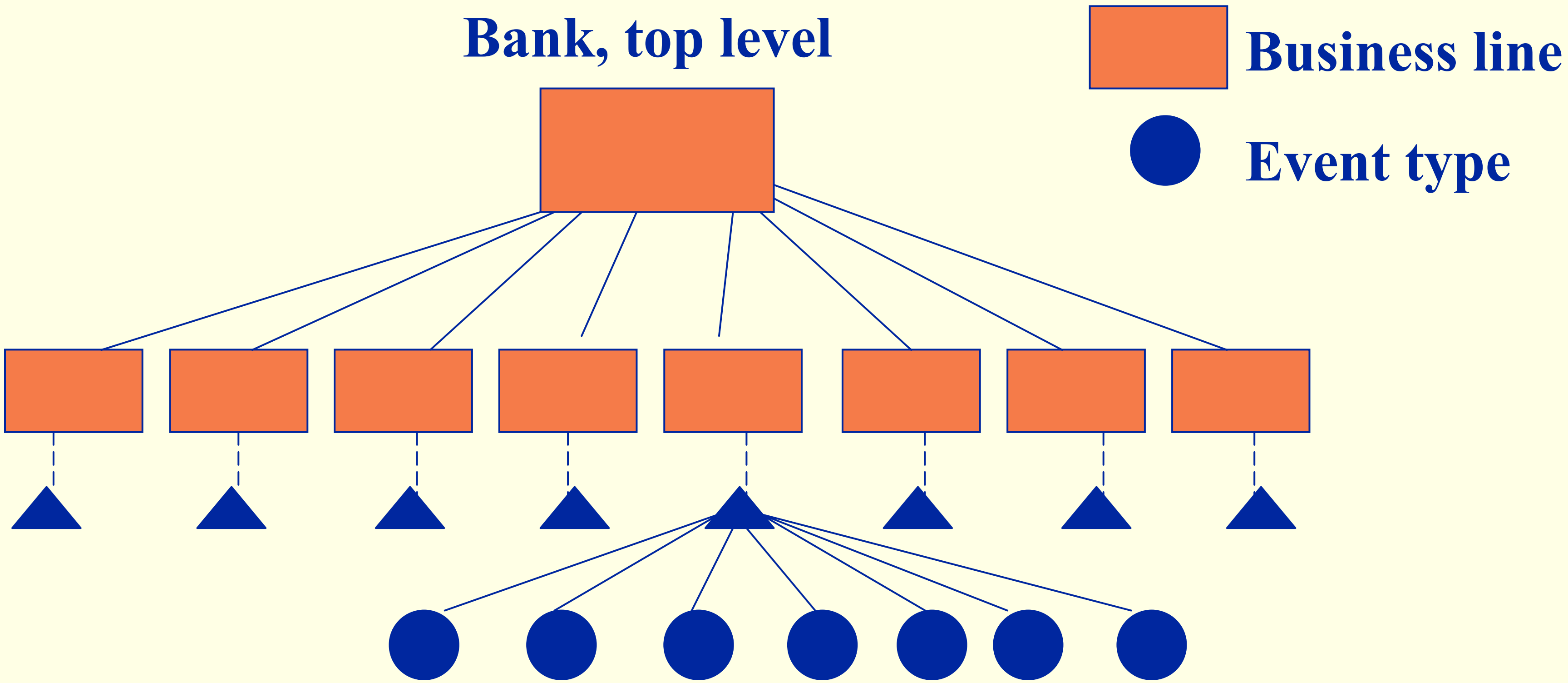
- **The Advanced Measurement Approaches (AMA)**
  Internal model for 56 risk cells (7 event types x 8 business line)

BCS has identified the following 7 risk event types:

- **Internal fraud:** e.g. intentional misreporting, employee theft, insider trading
- **External fraud:** e.g. robbery, cheque forgery, damage from computer hacking
- **Employment practices and workplace safety:** e.g. workers' compensation claims, violation of employee OH&S rules, union activities, discrimination claims
- **Clients, products and business practices:** e.g. misuse of confidential customer information, improper trading activities, money laundering, sale of unauthorised products.
- **Damage to physical assets:** e.g. terrorism, vandalism, earthquakes, fires, floods.
- **Business disruption and system failures:** e.g. hardware and software failures, telecommunication problems, utility outages, computer viruses.
- **Execution, delivery and process management:** e.g. data entry errors, management failures, incomplete legal documentation, unapproved access to client accounts

8 Business Lines

- Corporate finance(18%)
- Payment&Settlement(18%)
- Trading&Sales(18%)
- Agency Services(15%)
- Retail banking(12%)
- Asset management(12%)
- Commercial banking(15%)
- Retail brokerage(12%)


Bank, top level
Business line
Event type

Advanced Measurement Approach:

bottom-up Loss Distribution Approach

Insurance

Dependence Factors

Control/Risk Indicator

Internal Loss Data

(scaling, bias)

External Loss Data

(scaling, bias)

Expert Opinions

(business judgments on

#losses and $ranges)

Risk Frequency and
Severity distributions

Monte Carlo
simulations

Risk annual loss distributions

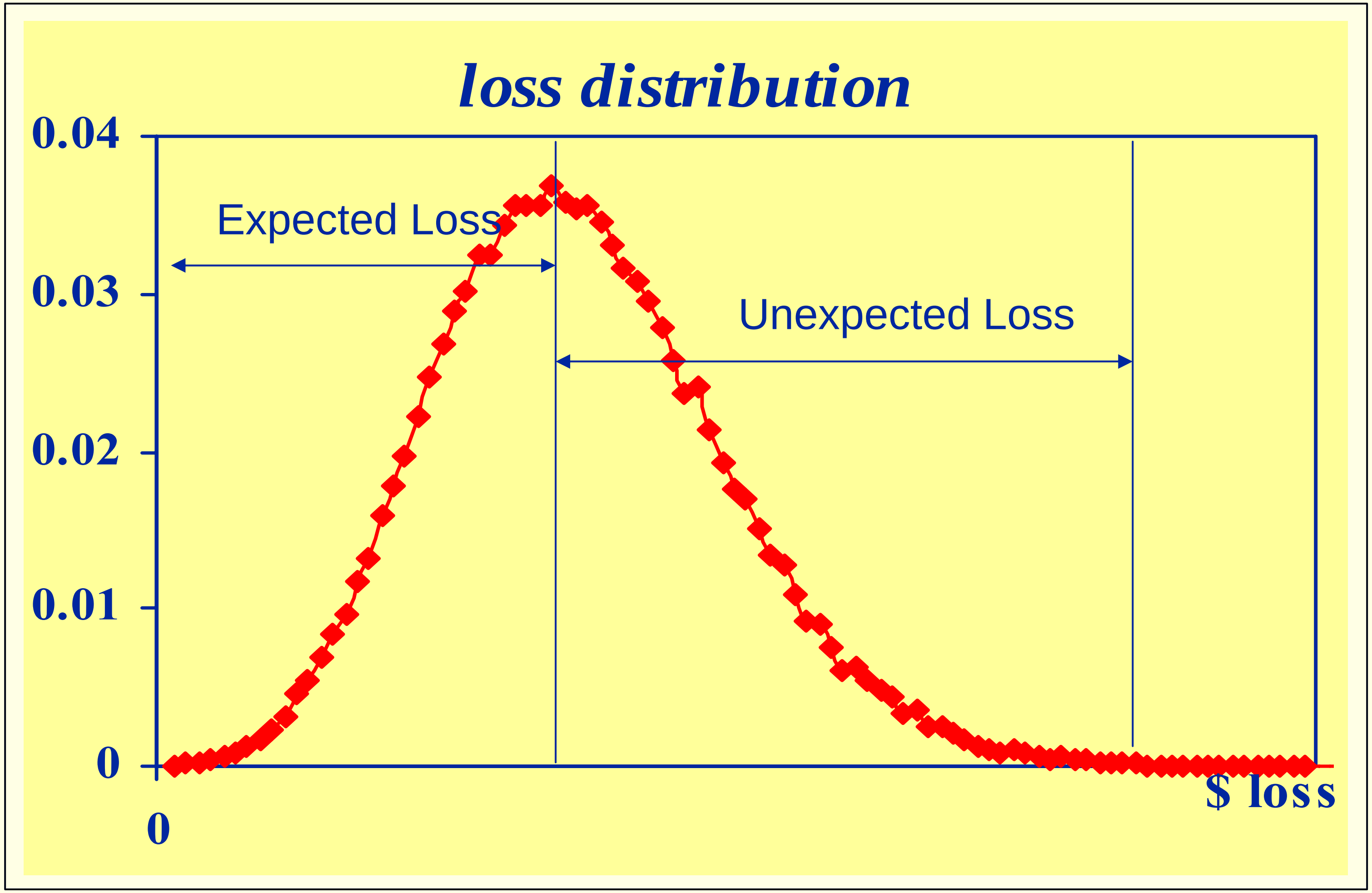
Total annual loss distribution (over all risks)

Annual Capital Charge (Expected and Unexpected losses)

Capital Allocation, what if scenarios

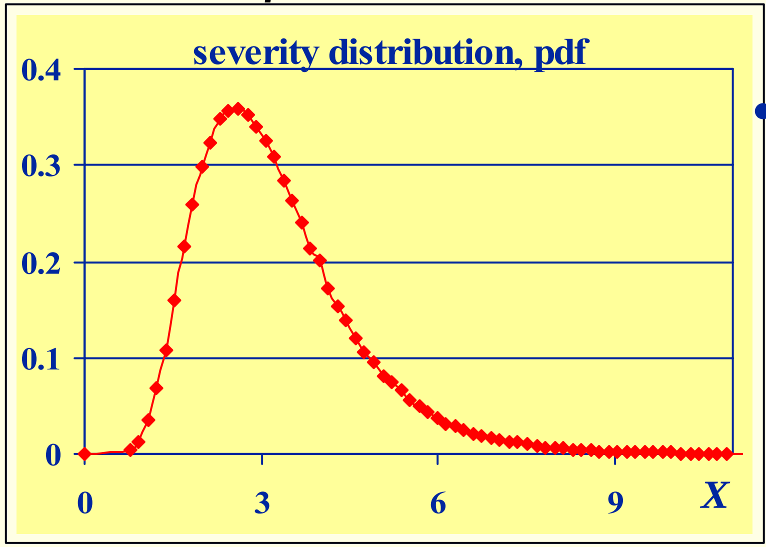
Annual Capital Charge

unexpected loss=$VaR_{0.999}$-Expected Loss; Pr [Loss$\leq VaR_{0.999}$]=0.999

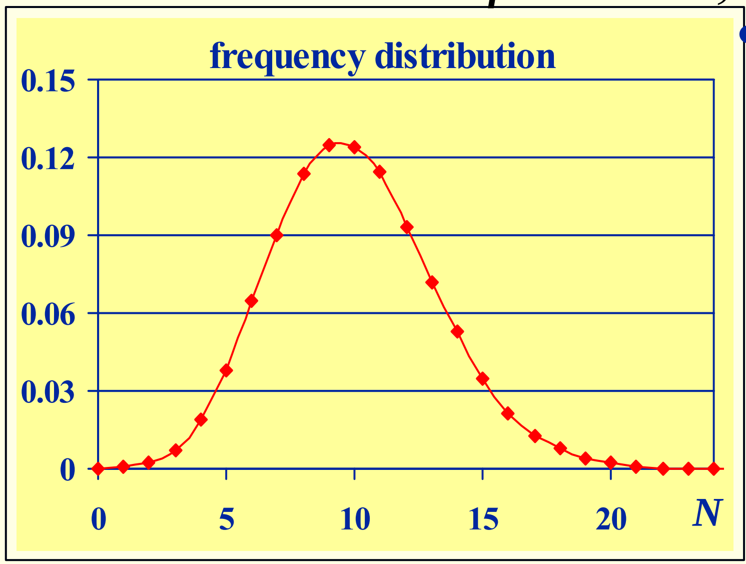


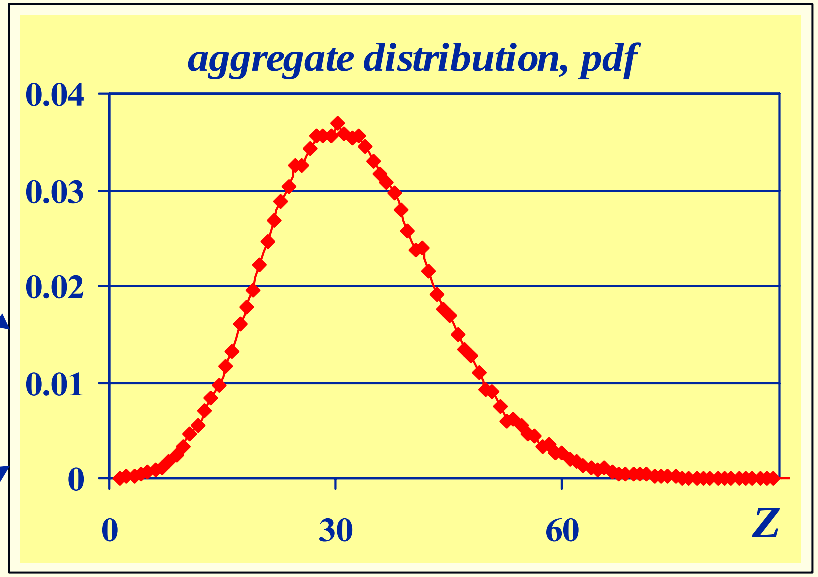
Loss Distribution Approach: single risk cell (business line/event type)

loss of the event, X



annual number of events, N





$$\text{annual loss}, Z = \sum_{k=1}^{N} X_k$$ **Monte Carlo, FFT**

$N \sim P(.|\boldsymbol{\theta})$, **e.g. Poisson**

$X_1, \ldots, X_N$ are iid $\sim f(.|\xi)$ **e.g. LogNormal**

N and $X_1, \ldots, X_N$ are independent

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Challenges/Tools

- Definition, identification, measurement, monitoring, indicators/controls
- Data Truncation: known threshold, stochastic threshold, unknown threshold
- ***Combining internal data, external data and expert opinions: credibility theory and Bayesian techniques***
- Data sufficiency: capital charge accuracy
- Correlation between risks and its estimation: copula, common shock processes
- Control indicators: regression/factorial analysis
- OR insurance: point processes
- Non-Gaussian distributions, Fat tails: EVT, mixed distributions, splices
- VaR pitfalls: coherent risk measures, expected shortfall
- Capital allocation

“[A] big challenge for us is how to mix the internal data with external data; this is something that is still a big problem because I don’t think anybody has a solution for that at the moment.” Or: “What can we do when we don’t have enough data [. . .] How do I use a small amount of data when I can have external data with scenario generation? [. . .] I think it is one of the big challenges for operational risk managers at the moment”.

An interview with four industry’s top risk executives in September 2006: Davis, E. (2006) Theory vs Reality. OpRisk and Compliance. 1 September 2006.

Combining internal data, industry data and expert opinions

- **Dominik Lambrigger (ETH), Pavel Shevchenko (CSIRO) and Mario Wüthrich (ETH).** ***The Quantification of Operational Risk using Internal Data, Relevant External Data and Expert Opinions.*** The Journal of Operational Risk 2(3), 3-27, 2007.
- **Hans Bühlmann (ETH), Pavel Shevchenko (CSIRO) and Mario Wüthrich (ETH).** ***A "Toy" Model for Operational Risk Quantification using Credibility Theory.*** The Journal of Operational Risk 2(1), 3-19, 2006.
- **Pavel Shevchenko (CSIRO) and Mario Wüthrich (ETH).** ***Structural Modelling of Operational Risk using Bayesian Inference: combining loss data with expert opinions.*** The Journal of Operational Risk 1(3), 3-26, 2006.
- **Pavel Shevchenko (CSIRO).** ***Estimation of Operational Risk Capital Charge under Parameter Uncertainty.*** The Journal of Operational Risk 3(1), 51-63, 2008.

Preprints are available from www.cmis.csiro.au/Pavel.Shevchenko

Combining internal data, industry data and expert opinions (SA)

Ad-hoc procedures: e.g.

internal data to estimate frequency and combined sample of

internal&external data to estimate severity;

Mixing distributions: $w_1 F_{SA}(X) + w_2 F_{\text{int}}(X) + (1 - w_1 - w_2) F_{ext}(X)$

Bayesian inference: combining two data sources – internal data and industry data (or SA)

observations $\mathbf{X} = (X_1, X_2, ..., X_n)$

parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_K)$

$$h(\mathbf{X}, \boldsymbol{\theta}) = h(\mathbf{X} \mid \boldsymbol{\theta}) \pi(\boldsymbol{\theta}) = \hat{\pi}(\boldsymbol{\theta} \mid \mathbf{X}) h(\mathbf{X})$$

$\hat{\pi}(\boldsymbol{\theta} \mid \mathbf{X}) \propto h(\mathbf{X} \mid \boldsymbol{\theta}) \pi(\boldsymbol{\theta})$ **posterior distribution, MCMC, Gaussian approx**

$\pi(\boldsymbol{\theta})$: **prior distribution is estimated by expert/industry data**

$h(\mathbf{X} \mid \boldsymbol{\theta})$: **likelihood of internal observations**

$\varphi(X_{n+1} \mid \mathbf{X}) = \int g(X_{n+1} \mid \boldsymbol{\theta}) \times \hat{\pi}(\boldsymbol{\theta} \mid \mathbf{X}) d\boldsymbol{\theta}$ **predictive distribution**

Modelling Frequency: conjugate priors (e.g. Poisson-Gamma)

$\mathbf{N} = (N_1, \ldots, N_n)$ are conditionally iid from $f(N \mid \lambda) = e^{-\lambda} \dfrac{\lambda^N}{N!}, \quad \lambda \geq 0$

Prior density: $\pi(\lambda \mid \alpha, \beta) = \dfrac{(\lambda / \beta)^{\alpha - 1}}{\Gamma(\alpha)\beta} \exp(-\lambda / \beta), \quad \lambda > 0, \alpha > 0, \beta > 0$

Posterior density:

$$\hat{\pi}(\lambda \mid \mathbf{N}) \propto \hat{\pi}(\lambda) h(\mathbf{N} \mid \lambda) = \frac{(\lambda / \beta)^{\alpha - 1}}{\Gamma(\alpha)\beta} \exp(-\lambda / \beta) \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{N_i}}{N_i!} \propto \lambda^{\hat{\alpha} - 1} \exp(-\lambda / \hat{\beta})$$

$$\alpha \to \hat{\alpha} = \alpha + \sum_{i=1}^{n} N_i,$$

$$\beta \to \hat{\beta} = \beta / (1 + \beta \times n).$$

Modelling Severity: conjugate priors (e.g. Lognormal-Normal, Pareto-Gamma, etc)

Non-informative constant priors to rely on internal data only

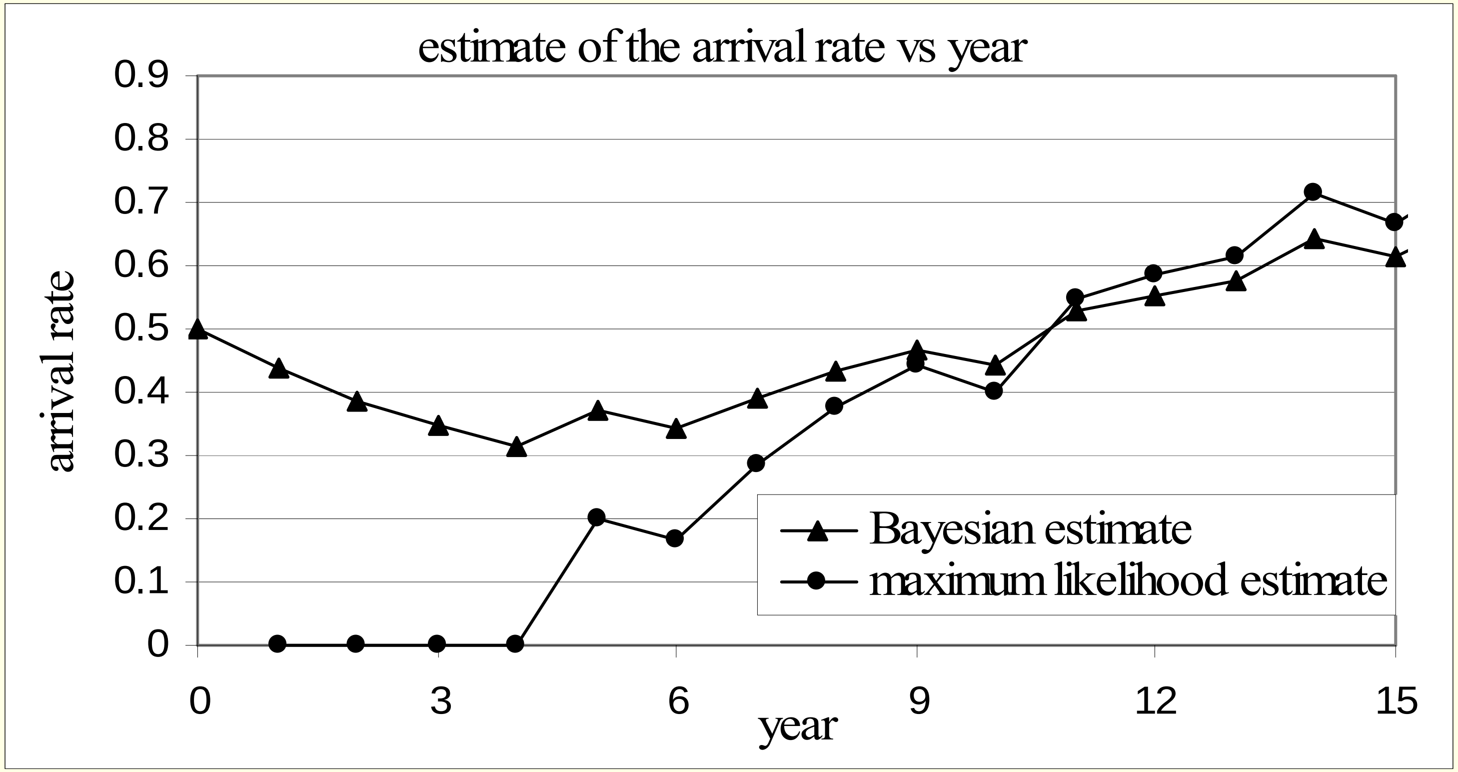
Example

Annual counts **N**=(0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 2, 1, 1, 2, 0) from Poisson $\lambda = 0.6$

expert opinions $E[\lambda] = 0.5$, $\Pr[0.25 \le \lambda \le 0.75] = 2/3 \Rightarrow \alpha \approx 3.41, \beta \approx 0.15$

$\hat{\lambda}_k = \hat{\alpha}_k \times \hat{\beta}_k$ the Bayesian estimator with Gamma prior $\alpha \approx 3.41, \beta \approx 0.15$

$\tilde{\lambda}_k = \frac{1}{k}\sum_{i=1}^{k} N_i$ the Maximum Likelihood estimator



Combining three data sources:

internal data, industry data and expert opinions

Bayesian inference

observations — expert opinions — parameters

$$\mathbf{X} = (X_1, X_2, \ldots, X_n) \qquad \boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, \ldots, \upsilon_M) \qquad \boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_K)$$

$$\hat{\pi}(\boldsymbol{\theta} \mid \mathbf{X}, \boldsymbol{\upsilon}) \propto h_1(\mathbf{X} \mid \boldsymbol{\theta}) h_2(\boldsymbol{\upsilon} \mid \boldsymbol{\theta}) \pi(\boldsymbol{\theta})$$ posterior: conjugate priors, MCMC, Gaussian approx

$\pi(\boldsymbol{\theta})$ prior distribution is estimated by industry data

$h_1(\mathbf{X} \mid \boldsymbol{\theta})$ likelihood of internal observations

$h_2(\boldsymbol{\upsilon} \mid \boldsymbol{\theta})$ likelihood of expert opinions

$$\varphi(X_{n+1} \mid \mathbf{X}) = \int g(X_{n+1} \mid \boldsymbol{\theta}) \times \hat{\pi}(\boldsymbol{\theta} \mid \mathbf{X}) d\boldsymbol{\theta}$$ predictive distribution

Modelling frequency: Poisson-Gamma-Gamma

$\mathbf{N} = (N_1, \ldots, N_n)$ **Annual counts are conditionally iid from** $f(N \mid \lambda) = e^{-\lambda} \dfrac{\lambda^N}{N!}, \quad \lambda \geq 0$

$\boldsymbol{\upsilon} = (\upsilon_1, \ldots, \upsilon_M)$ **Expert opinions are conditionally iid from** $Gamma(\xi, \lambda / \xi)$

$\pi(\lambda \mid \alpha, \beta) = \dfrac{(\lambda / \beta)^{\alpha - 1}}{\Gamma(\alpha)\beta} \exp(-\lambda / \beta), \quad \lambda > 0, \alpha > 0, \beta > 0$ **prior pdf from external data**

Posterior pdf, **Generalized Inverse Gamma** (GIG):

$$\hat{\pi}(\lambda \mid \mathbf{N}, \boldsymbol{\upsilon}) \propto \hat{\pi}(\lambda) h(\mathbf{N} \mid \lambda) h(\boldsymbol{\upsilon} \mid \lambda) = \frac{(\lambda / \beta)^{\alpha - 1}}{\Gamma(\alpha)\beta} \exp(-\lambda / \beta) \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{N_i}}{N_i!} \prod_{m=1}^{M} e^{-\upsilon_m \xi / \lambda} \frac{\upsilon_m^{\xi - 1}}{(\lambda / \xi)^{\xi}}$$

$$\propto \lambda^{\nu} \exp(-\lambda\omega - \phi / \lambda)$$

$$\nu = \alpha - 1 + \sum_{i=1}^{n} N_i - M\xi,$$

$$\omega = n + 1 / \beta$$

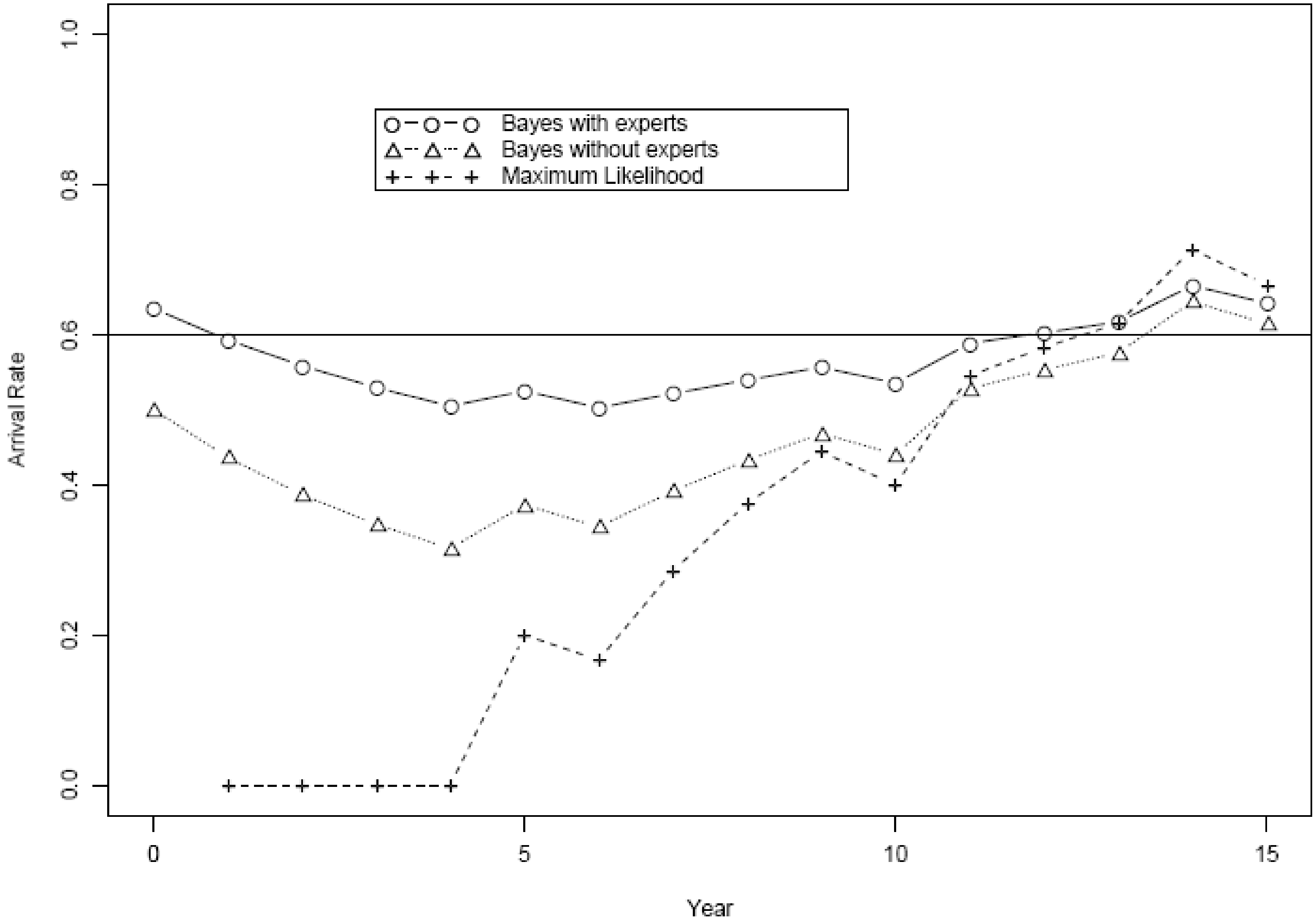
$$\phi = \xi \times \sum_{m=1}^{M} \upsilon_m .$$

Modelling severity: Pareto-Gamma-Gamma, Lognormal-Normal-Normal

Example: **Annual counts** **N**=(0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 2, 1, 1, 2, 0) from Poisson(0.6)

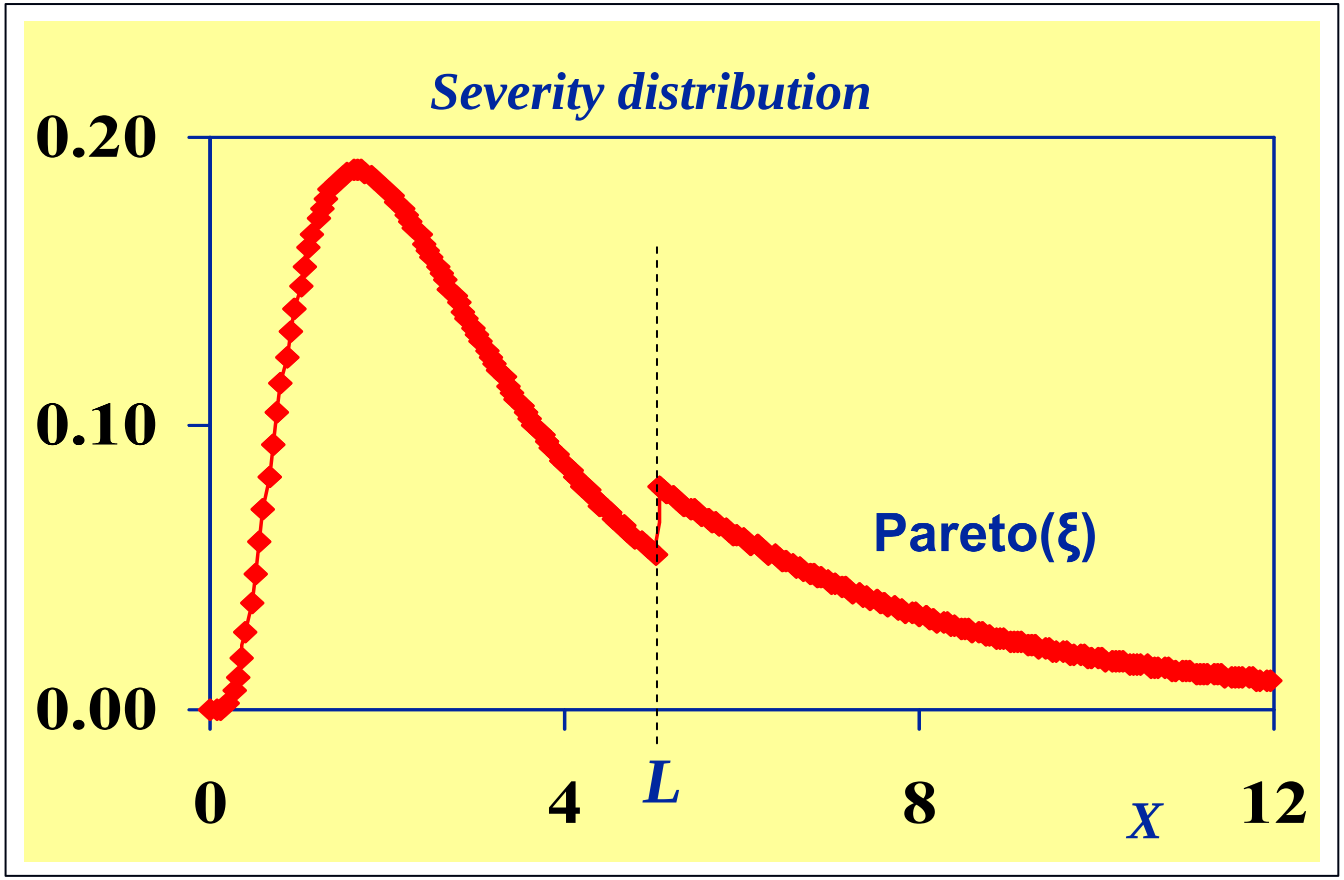
External data $E[\lambda]=0.5, \Pr[0.25 \le \lambda \le 0.75]=2/3 \Rightarrow \alpha \approx 3.41, \beta \approx 0.15$

Expert $\hat{\upsilon}=0.7, Vco(\upsilon \mid \lambda)=0.5$

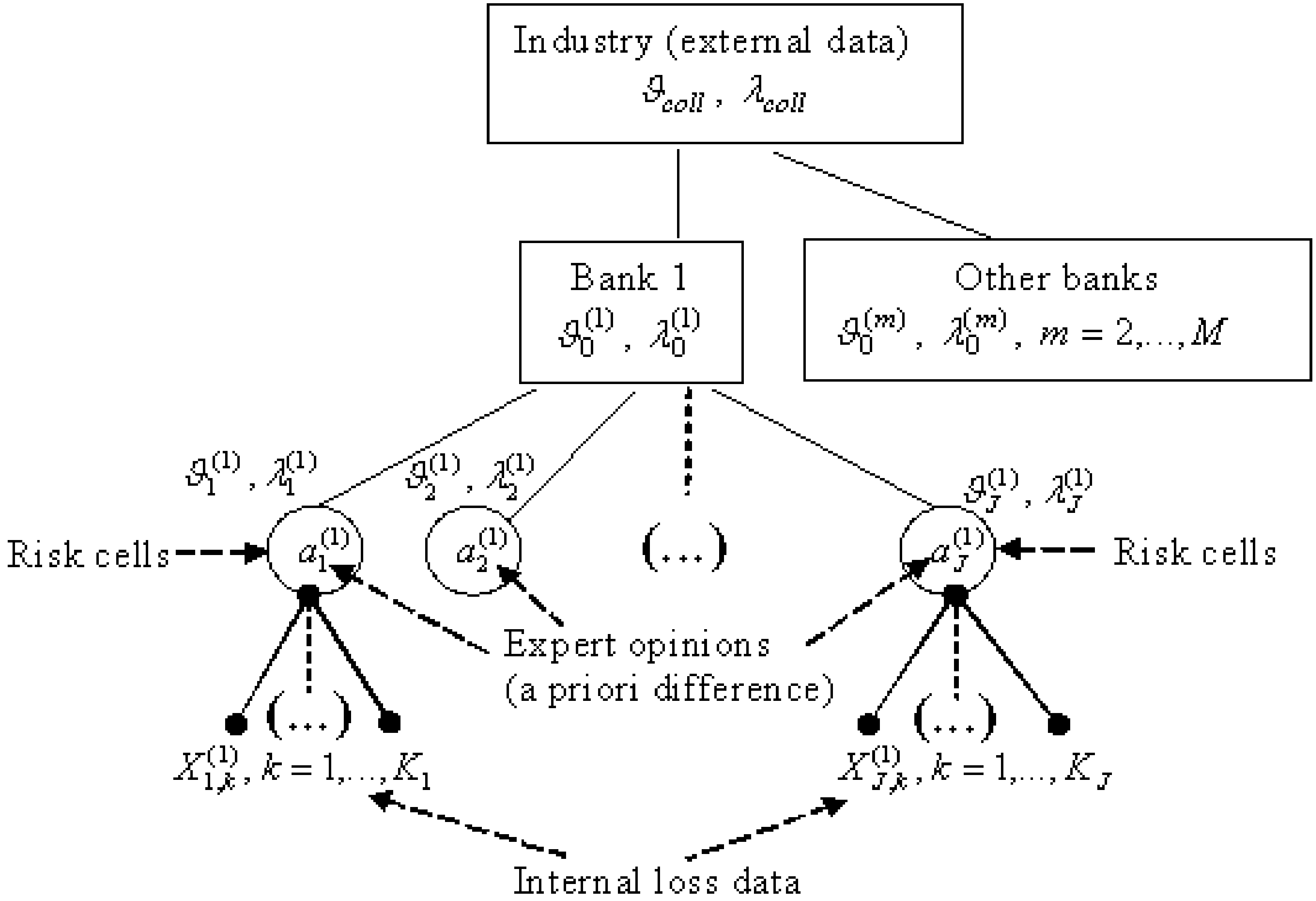


"Toy" model for Low Frequency High Impact Losses using credibility theory: Poisson-Pareto

$$f(x|\xi) = \frac{\xi}{L}\left(\frac{x}{L}\right)^{-\xi-1}, x \geq L, \xi > 0; \quad P(n|\theta) = \frac{\theta^n}{n!}e^{-\theta}, \; n = 0,1,\ldots, \; \theta \geq 0$$



$$Z_j = \sum_{n=1}^{N_j} X_{j,n} \Leftarrow X_{j,n} \sim \text{Pareto}(\hat{\hat{\xi}}_j = a_j \hat{\hat{\vartheta}}_j), \quad N_j \sim \text{Poisson}(\hat{\hat{\theta}}_j = \nu_j \hat{\hat{\lambda}}_j)$$



Credibility BS model : portfolio of J risks with observations $Y_{j,k} : k = 1,\dots,K_j$. Assume that for known weights $w_{j,k}$, θ_j (realization of rv Θ_j) is a risk profile of the j-th risk and

$a)$ $Y_{j,k} : k = 1,\dots,K_j$ are conditionally independent with

$$E[Y_{j,k} \mid \Theta_j] = \mu(\Theta_j),\ \text{var}[Y_{j,k} \mid \Theta_j] = \sigma^2(\Theta_j)/w_{j,k}$$

$b)$ $(\Theta_1, \mathbf{Y}_1),\dots,(\Theta_J, \mathbf{Y}_J)$ are independent

$c)$ $\Theta_1,\dots,\Theta_J$ are iid

Define : $\mu_0 = E[\mu(\Theta_j)], E[\sigma^2(\Theta_j)] = \sigma^2, \text{var}[\mu(\Theta_j)] = \tau^2$

Homogeneous credibility estimator :

$$\hat{\mu}(\Theta_j) = \alpha_j Y_j + (1-\alpha_j)\hat{\mu}_0,$$

$$\hat{\mu}_0 = \sum_{j=1}^{J} \frac{\alpha_j}{\alpha_0} Y_j, Y_j = \sum_{k=1}^{K_j} \frac{w_{j,k}}{w_{j,\bullet}} Y_{j,k}, \alpha_j = \frac{w_{j,\bullet}}{w_{j,\bullet} + \sigma^2/\tau^2}, \alpha_0 = \sum_{j=1}^{J} \alpha_j$$

Parameter Risk (uncertainty of parameters)

$$\mathbf{Y} = (\mathbf{X}, \mathbf{N}) - \text{past} \quad \text{observations}; \quad Z = \sum_{i=1}^{N} X_i - \text{annual loss}$$

$$\varphi(Z_{t+1} \mid \mathbf{Y}) = \int g(Z_{t+1} \mid \boldsymbol{\theta}) \times \hat{\pi}(\boldsymbol{\theta} \mid \mathbf{Y}) d\boldsymbol{\theta} - \text{predictive distribution}$$

$$\hat{Q}_{0.999}^{B} - 0.999 \text{ quantile of} \quad \varphi(Z_{t+1} \mid \mathbf{Y})$$

$$\hat{Q}_{0.999} - 0.999 \text{ quantile of} \quad g(Z_{t+1} \mid \hat{\boldsymbol{\theta}})$$

$\hat{\boldsymbol{\theta}}$ is point estimator, e.g. maximum likelihood estimator

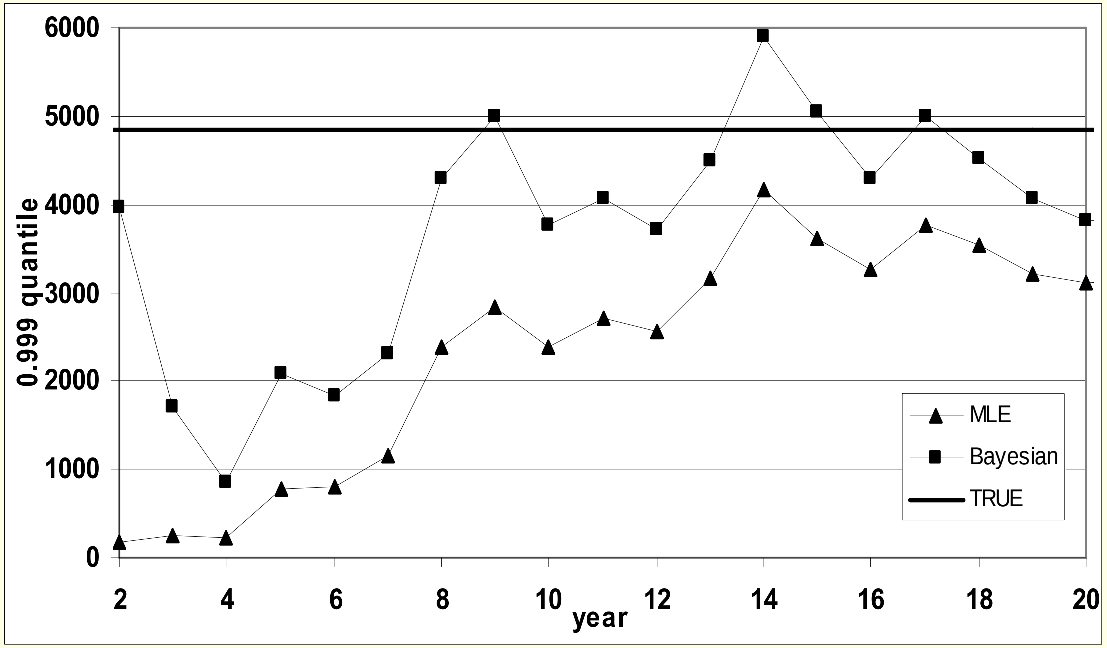
$$\hat{\pi}(\boldsymbol{\theta} \mid \mathbf{Y}) \propto h(\mathbf{Y} \mid \boldsymbol{\theta}) \pi(\boldsymbol{\theta})$$

$$bias = E[\hat{Q}_{0.999}^{B} - \hat{Q}_{0.999}]$$

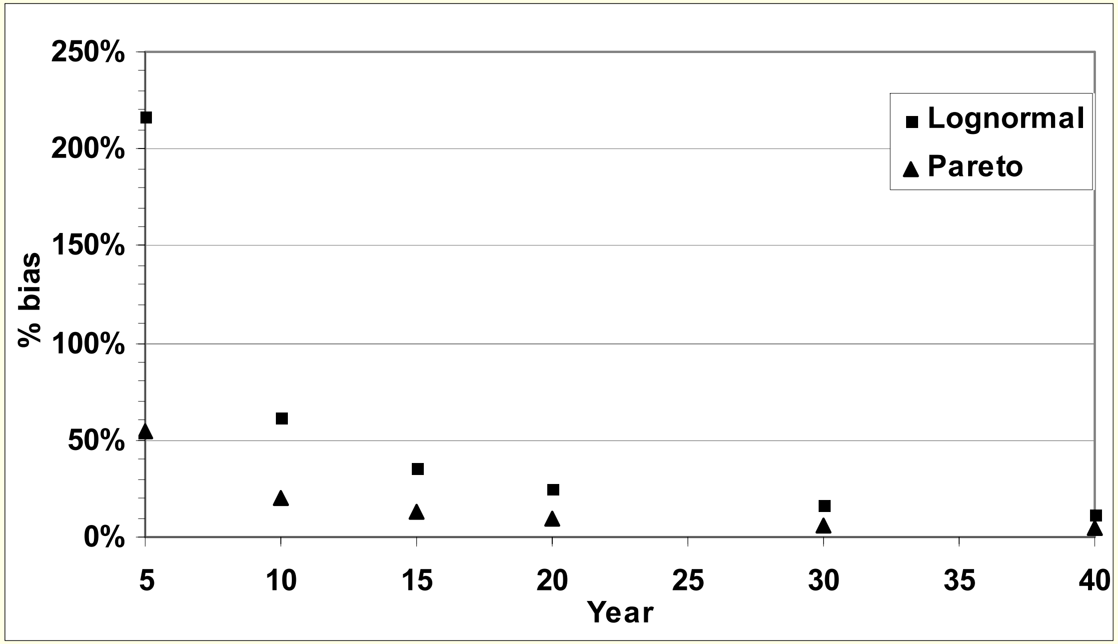
$$\text{Normal approximation}: \hat{\pi}(\hat{\boldsymbol{\theta}} \mid \mathbf{Y}) \text{ is } Normal(mean = \hat{\boldsymbol{\theta}}, \text{cov} = -\mathbf{I}^{-1})$$

$$\ln \hat{\pi}(\boldsymbol{\theta} \mid \mathbf{Y}) \approx \ln \hat{\pi}(\hat{\boldsymbol{\theta}} \mid \mathbf{Y}) + \frac{1}{2} \sum_{i,j} \mathbf{I}_{ij} (\theta_i - \hat{\theta}_i)(\theta_j - \hat{\theta}_j); \quad \mathbf{I}_{ij} = \left. \frac{\partial^2 \ln \hat{\pi}(\boldsymbol{\theta} \mid \mathbf{Y})}{\partial \theta_i \partial \theta_j} \right|_{\theta = \hat{\theta}}$$

Parameter Risk (uncertainty of parameters)



Comparison of the 0.999 quantile estimators. Parameter uncertainty is ignored by (MLE) but is taken into account by (Bayesian). MLE - maximum likelihood estimator Bayesian - quantile of predictive distribution Losses were simulated from *Poisson*(10) and *LN*(1,2). Non-informative constant prior were used.



Relative bias in the 0.999 quantile estimator induced by the parameter uncertainty vs number of observation years. (Lognormal) - losses were simulated from *Poisson*(10) and *LN*(1,2). (Pareto) – losses were simulated from *Poisson*(10) and *Pareto*(2) with *L*=1.

Topics of current research in operational risk

- Evolutionary models (stochastic risk profiles)
- Dependence between risks via dependence between risk profiles
- MCMC (Metropolis-Hastings within Gibbs random walk, Slice Sampling) to calibrate models
- Impact of data truncation
- Numerical methods to calculate compound process distributions using characteristics functions (direct integration, FFT)

Collaborators/industry clients: ETH Zurich, Vienna University of Technology, Statistical Research Associates NZ, Commonwealth Bank of Australia, Australia New Zealand bank

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Thank you

Maximum Likelihood Estimator for tail parameter:

Consider risk cells $j = 1,\ldots,J$ with losses $X_{j,k} \geq L,\ k = 1,\ldots,K_j$

$X_{j,k}$ are (conditionally) iid from Pareto($\xi_{\mathrm{j}} = a_j \vartheta_j$)

Then the "maximum likelihood estimator"

$$\hat{\vartheta}_j = \left[\frac{a_j}{K_j - 1} \sum_{k=1}^{K_j} \ln\left(\frac{X_{j,k}}{L} \right) \right]^{-1},$$

$$E[\hat{\vartheta}_j \mid \vartheta_j] = \vartheta_j,\ Var[\hat{\vartheta}_j \mid \vartheta_j] = \frac{\vartheta_j^2}{K_j - 2},$$

$$\hat{\xi}_j = a_j \hat{\vartheta}_j$$

Improved credibility estimator (using all data in the bank):

Assume ϑ_j, $j = 1,...,J$ are iid with $\mathrm{E}[\vartheta_j] = \vartheta_0$, $Var[\vartheta_j] = \tau_0^2$

ϑ_0 is a risk profile of the bank. Then

$$\hat{\hat{\vartheta}}_j = \alpha_j \hat{\vartheta}_j + (1-\alpha_j)\vartheta_0, \text{ where } \quad \alpha_j = \frac{K_j - 2}{K_j - 1 + (\vartheta_0 / \tau_0)^2}$$

$$\hat{\tau}_0^2 = \frac{1}{J \; -1} \sum_{j=1}^{J} \alpha_j (\hat{\vartheta}_j - \hat{\vartheta}_0)^2,$$

$$\hat{\vartheta}_0 = \frac{1}{W} \sum_{j=1}^{J} \alpha_j \hat{\vartheta}_j, \quad W \; = \sum_{j=1}^{J^{(1)}} \alpha_j$$

$$\hat{\hat{\xi}}_j = a_j \hat{\hat{\vartheta}}_j, \; j = 1,...,J$$

Improved credibility estimator (using industry data):

Consider M banks with risk profiles $\vartheta_0^{(m)}, m = 1,\ldots,M$

Assume that $\vartheta_0^{(m)}, m = 1,\ldots,M$ are iid with $\mathrm{E}[\vartheta_0^{(m)}] = \vartheta_{\mathrm{coll}}, Var[\vartheta_0^{(m)}] = \tau_{coll}^2$

$\vartheta_{\mathrm{coll}}$ is a risk profile of the industry. Then

$$\hat{\vartheta}_j^{(m)} = \left[\frac{a_j^{(m)}}{K_j^{(m)} - 1} \sum_{k=1}^{K_j^{(m)}} \ln\left(\frac{X_{j,k}^{(m)}}{L^{(m)}} \right) \right]^{-1}, \quad \alpha_j^{(m)} = \frac{K_j^{(m)} - 2}{K_j^{(m)} - 1 + \left(\frac{\vartheta_0^{(m)}}{\tau_0^{(m)}} \right)^2}, \quad \begin{pmatrix} j = 1,\ldots,J^{(m)} \\ m = 1,\ldots,M \end{pmatrix}$$

$$\hat{\vartheta}_0^{(m)} = \frac{1}{W^{(m)}} \sum_{j=1}^{J^{(m)}} \alpha_j^{(m)} \hat{\vartheta}_j^{(m)}, \quad \beta^{(m)} = \frac{W^{(m)}}{W^{(m)} + \left(\frac{\tau_0^{(m)}}{\tau_{coll}} \right)^2}, \quad W^{(m)} = \sum_{j=1}^{J^{(m)}} \alpha_j^{(m)}, m = 1,\ldots,M$$

$$\hat{\vartheta}_{coll} = \frac{1}{A} \sum_{j=1}^{J^{(m)}} \beta^{(m)} \hat{\vartheta}_0^{(m)}, \quad A = \sum_{m=1}^{M} \beta^{(m)}$$

Corrected credibility estimators top - down

$$\hat{\hat{\vartheta}}_0^{(1)} = \beta^{(1)}\hat{\vartheta}_0^{(1)} + (1-\beta^{(1)})\hat{\vartheta}_{coll},$$

$$\hat{\hat{\vartheta}}_j^{(1)} = \alpha_j^{(1)}\hat{\vartheta}_j^{(1)} + (1-\alpha_j^{(1)})\hat{\hat{\vartheta}}_0^{(1)},\ j=1,\ldots,J^{(m)} \Rightarrow \hat{\hat{\xi}}_j^{(1)} = a_j^{(1)}\hat{\hat{\vartheta}}_j^{(1)}$$

Industry structural parameters

$$\hat{\tau}_{coll}^2 = \max\left[c\times\left\{\frac{M}{M-1}\sum_{m=1}^{M}\frac{W^{(m)}}{W_0}(\hat{\vartheta}_0^{(m)} - \overline{\hat{\vartheta}_0^{(m)}})^2 - \frac{M\hat{\tau}^2}{W_0}\right\}, 0\right]$$

$$\hat{\tau}^2 = \frac{1}{M}\sum_{m=1}^{M}(\hat{\tau}_0^{(m)})^2, W_0 = \sum_{m=1}^{M}W^{(m)}, \overline{\hat{\vartheta}_0^{(m)}} = \frac{1}{M}\sum_{m=1}^{M}\hat{\vartheta}_0^{(m)},$$

$$c = \frac{M-1}{M}\left\{\sum_{m=1}^{M}\frac{W^{(m)}}{W_0}\left(1-\frac{W^{(m)}}{W_0}\right)\right\}^{-1}$$

Maximum Likelihood Estimator for arrival rate:

Consider risk cells $j = 1,\dots,J$ with loss frequencies $N_{j,k}$, $k = 1,\dots,K_j$

$N_{j,k}$ are (conditionally) iid from Poisson($\theta_j = \nu_j \lambda_j$)

ν_j are known constants and λ_j are risk profiles

Then the maximum likelihood estimator

$$\hat{\lambda}_j = \frac{1}{\tilde{\nu}_j} \sum_{k=1}^{K_j} N_{j,k}, \quad \tilde{\nu}_j = \nu_j K_j$$

$$E[\hat{\lambda}_j \mid \lambda_j] = \lambda_j, \quad \mathrm{Var}[\hat{\lambda}_j \mid \lambda_j] = \lambda_j / \tilde{\nu}_j,$$

Improved credibility estimator for arrival rate (using bank data)

Assume $\lambda_j,\ j=1,...,J$ are iid with $\mathrm{E}[\lambda_j]=\lambda_0, Var[\lambda_j]=\omega_0^2$

λ_0 is a risk profile of the bank.

Consider $F_{j,k}=N_{j,k}/\nu_j, E[F_{j,k}]=\lambda_j, \mathrm{Var}[F_{j,k}]=\lambda_j/\nu_j$ then

$$\hat{\hat{\lambda}}_j=\gamma_j\hat{\lambda}_j+(1-\gamma_j)\lambda_0,\ \text{where}\ \gamma_j=\frac{\tilde{\nu}_j}{\tilde{\nu}_j+\lambda_0/\omega_0^2}\quad\Rightarrow\hat{\hat{\theta}}_j=\nu_j\hat{\hat{\lambda}}_j$$

$$\hat{\omega}_0^2=\max\left[c\times\left\{T-\frac{J\ \hat{\lambda}_0}{\nu_0}\right\},0\right];\quad \hat{\lambda}_0=\frac{1}{\tilde{\gamma}}\sum_j\gamma_j\hat{\lambda}_j,\tilde{\nu}_j=\nu_jK_j$$

$$\nu_0=\sum_{j=1}^{J}\tilde{\nu}_j;\ T=\frac{J}{J-1}\sum_{j=1}^{J}\frac{\tilde{\nu}_j}{\nu_0}(\hat{\lambda}_j-\overline{F})^2;\ \tilde{\gamma}=\sum_j\gamma_j;$$

$$\overline{F}=\frac{1}{J}\sum_{j=1}^{J}\hat{\lambda}_j;\ c=\frac{J}{J-1}\left\{\sum_{j=1}^{J}\frac{\tilde{\nu}_j}{\nu_0}\left(1-\frac{\tilde{\nu}_j}{\nu_0}\right)\right\}^{-1}.$$

Improved credibility estimator of arrival rate (using industry data)

Consider M banks with risk profiles $\lambda_0^{(m)}, m = 1,\ldots,M$

Assume that $\lambda_0^{(m)}, m = 1,\ldots,M$ are iid with $\mathrm{E}[\lambda_0^{(m)}] = \lambda_{\mathrm{coll}}, Var[\lambda_0^{(m)}] = \omega_{coll}^2$

λ_{coll} is a risk profile of the industry. Then

$$\hat{\lambda}_j^{(m)} = \frac{1}{\tilde{v}_j^{(m)}} \sum_{k=1}^{K_j^{(m)}} N_{j,k}^{(m)}, \quad \gamma_j^{(m)} = \frac{\tilde{v}_j^{(m)}}{\tilde{v}_j^{(m)} + \lambda_0^{(m)} / (\omega_0^{(m)})^2}, \; j = 1,\ldots,J^{(m)}, m = 1,\ldots,M$$

$$\hat{\lambda}_0^{(m)} = \frac{1}{W^{(m)}} \sum_{j=1}^{J^{(m)}} \gamma_j^{(m)} \hat{\lambda}_j^{(m)}, \; \rho^{(m)} = \frac{W^{(m)}}{W^{(m)} + \left(\dfrac{\omega_0^{(m)}}{\omega_{coll}}\right)^2}, \; W^{(m)} = \sum_{j=1}^{J^{(m)}} \gamma_j^{(m)}, m = 1,\ldots,M$$

$$\hat{\lambda}_{coll} = \frac{1}{A} \sum_{j=1}^{J^{(m)}} \rho^{(m)} \hat{\lambda}_0^{(m)}, \quad A = \sum_{m=1}^{M} \rho^{(m)}$$

Corrected credibility estimators for arrival rate: top - down

$$\hat{\hat{\lambda}}_0^{(1)} = \rho^{(1)}\hat{\lambda}_0^{(1)} + (1-\rho^{(1)})\hat{\lambda}_{coll},$$

$$\hat{\hat{\lambda}}_j^{(1)} = \gamma_j^{(1)}\hat{\lambda}_j^{(1)} + (1-\gamma_j^{(1)})\hat{\hat{\lambda}}_0^{(1)}, j = 1,\ldots, J^{(m)}$$

$$\Downarrow$$

$$\hat{\hat{\theta}}_j^{(1)} = \nu_j^{(1)}\hat{\hat{\lambda}}_j^{(1)}$$