Brennan and Schwartz (1982) model

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Brennan and Schwartz (1982) introduced two factor interest rate model where the actual (real world) processes of the short r_t and long χ_t rates have the form:

$$dr_t = (\alpha_1 + \beta_1(\chi - r_t))dt + \sigma_1 dX_t$$
$$d\chi_t = \chi_t(\alpha_2 + \beta_2 r + \gamma\chi_t)dt + \sigma_2 dW_t$$

Here, X_t and W_t are correlated standard Brownian processes with correlation coefficient ρ and α_1, β_1, σ_1, α_2, β_2, γ, σ_2 are constant parameters. The model assumes that the short and the long rates correspond to the observed spot and consol rates. The later is the yield on the irredeemable bond, so-called consol bond, which is a fixed coupon bond of infinite maturity. If the market price of the consol bond is C_0 and annual coupon is \$1 then the yield (consol rate) will satisfy $\chi = 1/C_0$. Note that, the consol bond is traded security and thus the consol rate is observable variable.

In general, one-factor models consider one source of uncertainty that affects all rates of the yield curve and implies that the whole term structure of interest rates can be inferred from the current instantaneous rate. It does not imply that the yield curve will move in parallel as the rates can be affected by driving variable to a different extend, but implies perfect correlation between movements of the rates with different maturities. However, empirical observations suggest that correlation between the rates decreases exponentially as distance between maturities increases. Two factor models are less ambitious than one factor models and assume that the whole term structure of the interest rates can be explained by two variables. In particular, the model of Brennan and Schwartz uses long and instantaneous rates to derive intermediate part of the curve in terms of these extremities, while one factor models derive the long term rate as deterministic function of the instantaneous rate. The rationale for the introduction of a second factor is to induce decoupling between forward market rates sufficient to recover market prices of the instruments that depend on different parts of the yield curve (e.g. swaptions). Note that, one factor models can be appropriate for pricing of instruments that depend on the level of yield curve only, for example, vanilla options.

Two factor interest rate models allow for more realistic pricing but, in general, two utility functions (market prices of risks) associated to the model variables have to be introduced. The achievement of the methodology proposed by Brennan and Schwartz is that their model requires market price of risk for the short rate only. This is because the second variable is taken as the consol rate which is inversely proportional to the consol bond price and the risk associated to the variable can be hedged away. Under the general process for the short and long rates

$$dr_t = \mu_r(r,\chi,t)dt + \sigma_r(r,\chi,t)dX_t$$
$$d\chi_t = \mu_\chi(r,\chi,t)dt + \sigma_\chi(r,\chi,t)dW_t$$

the market price of risk λ_χ associated with the consol rate will satisfy

$$\lambda_\chi = (r\chi - \chi^2 + \mu_\chi)/\sigma_\chi - \sigma_\chi/\chi$$

This general result allows for elimination of the consol market risk from the bond pricing equation corresponding to the Brennan and Schwartz model.

The model specific parametrisation implies locally log-normal behavior for the rates. Also, the drift part of the short rate has mean reverting form implying reversion of the short rate to the long rate with reversion speed β_1. The deterministic component of the consol rate comes from the assumption that the market price of the risk associated with the consol rate is linear in rates, that is $\lambda_\chi = \theta_1 + \theta_2 r + \theta_3 \chi$.

Brennan and Schwartz used real data for US rates over the period 1958-1979 to estimate model parameters α_1, β_1, σ_1, α_2, β_2, γ, σ_2 and ρ statistically. Then, the market price of risk associated with the short rate λ_r was estimated from bond prices. Note that, zero coupon bond equation for the model has no closed-form solution and numerical method (in particular, two dimensional finite difference method) had to be used.

The specific parametric form of the model has a number of problems. For example, negative and sufficiently large α_1 may cause negative short rate over finite period of time. The joint dynamics of the short and long rates can be unstable. The long rate may explode (go to infinity) in finite time as demonstrated by Hogan (1993). The model exploding phenomena will be immediately encountered if one will try to simulate the rates using Monte Carlo method. In the risk neutral world, the consol bond price can be calculated as the expectation of the payoffs discounted along the short rate paths, and thus the model parameters should satisfy some constraints to ensure internal consistency between the short and consol rate processes. Duffie, Ma and Yong (1994) demonstrated that the consol and short rate processes can be compatible for a particular choice of volatility. Also, due to the relatively complicated parametric form of the model, bond pricing equation has no closed form solution and thus should be solved numerically using, for example, finite difference or lattice methods. It is important to note that these methods can be quite complicated to handle path dependent options in the case of two factor models.

The original work of Brennan and Schwartz (1982) was to develop equilibrium model for bond pricing to test market efficiency. Although specific parametric form of the model used by Brennan and Schwartz suffers from a number of problems, the general framework suggested is appealing. The framework can be extended to overcome the above mentioned shortcomings and to develop a no-arbitrage models explored in Rebonatto (1998).

Further Reading

Brennan, M. and E. Schwartz (1982), An Equilibrium model of Bond Pricing and a Test of Market Efficiency, *Journal of Financial and Quantitative Analysis*, 17, 3, 301-329.

Rebonato, R. (1998), *Interest-Rate Option Models*, 2nd edition, John Wiley & Sons, p. 341-352.

Hogan, M. (1993), Problems in certain two-factor term structure models. *Annals of Applied Probability* 3, 576-591.

Duffie, D., J.Ma, and J.Yong (1994), Black's consol rate conjecture, working paper, Graduate School of Business, Stanford University, Stanford, CA.

See also: Interest rate derivatives; Yield curve; Cox-Ingersoll-Ross (CIR) interest rate model (1985); Fong & Vasicek (1991) model; Hull-White (1987) interest rate model; Yield curve modelling/fitting; Swaptions.