DECOMPOSITION OF THE KUWAHARA-NAGAO OPERATOR IN TERMS OF A LINEAR SMOOTHING AND A MORPHOLOGICAL SHARPENING

R. VAN DEN BOOMGAARD
Intelligent Sensory Information Systems
University of Amsterdam
The Netherlands

Abstract This report sheds some new light on an old and trusted tool in the image processing toolbox. The Kuwahara-Nagao operator is known as an edge preserving smoothing operator.

This report shows that we don't need to trust our intuition to verify this claim but that it follows from basic principles. To that end we will first cast the classical Kuwahara-Nagao operator into a more modern framework using Gaussian convolutions to calculate the mean and variance of local image patches and using morphological flow fields to select mean value of the neighborhood that has minimal variance.

The main result of this report is the formulation of the PDE whose solution is equivalent to the iteration of a Kuwahara-Nagao operator using infinitesimally small neighborhoods. The resulting PDE is a combination of linear diffusion and morphological sharpening.

Keywords: Edge preservation, PDEs, smoothing operator.

1. Introduction

The classical Kuwahara operator [1] uses four neighborhoods of a point (k, l) on the sampling grid as depicted in Fig. 1. Within each of the neighborhoods both the mean and variance are calculated. The mean of the neighborhood that has the lowest variance is taken as the new value for the point (k, l). The Nagao [2] operator uses all 3×3 neighborhoods that fit within a 5×5 neighborhood. Extensions to larger neighborhoods can be found in literature. In Fig. 2 some examples of the Kuwahara-Nagao operator are shown.

Schulze and Pearce [3] describe the Kuwahara operator within the frame work of what they called *the value-and-criterion filter structure*. They also noted the link with morphological operators. Their operator structure is the starting point in defining *morphological flow fields* in Section 2 and generalizing

Redistribution rights reserved CSIRO Publishing. ISBN 0 643 06804 X

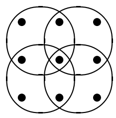

Figure 1. **Kuwahara Image Operator.** The new value of the central pixel is calculated as the mean of the region (one of the 4 depicted regions) that has minimal variance.

the Kuwahara-Nagao operator to work with Gaussian convolutions instead of uniform convolutions (see Section 3).

For the modern eye the result of the Kuwahara-Nagao operator is reminiscent of the result obtained with sharpening-smoothing operators that are well-known since the introduction of PDE based image operators. Sharpening-smoothing PDE controlled image evolution processes date back to the sixties (see Gabor [4] and Lindenbaum et al. [5]). There is a lot of modern literature on the subject too (see [6, 7, 8, 9, 10]). In Section 4 we show that the generalized Kuwahara-Nagao operator as introduced in Section 3 is equivalent to a smoothing-sharpening PDE operator in the limit case of infinitesimally small scale (of the Gaussian kernels involved and of the erosion structuring element).

2. Morphological Flow Fields

Consider the erosion operator ϵ in the function space $Fun(E,V)$. It has been shown [11] that any erosion ϵ in $Fun(E,V)$ (with an ordering that is defined in V and 'lifted' to $Fun(E,V)$) can be written as:

$$\epsilon(f)(x) = \bigwedge_{y \in E} e_{y,x}(f(y)),$$

where $e_{y,x}$ is a two parameter family of erosions in the complete lattice $(V, \leq)$.

Given the above representation of the erosion ϵ we define the *erosion flow operator* $\vec{\epsilon}$ to result in the flow field that at each position defines the relative vector pointing to (one of) the points where the infimum is attained.

Definition 1 (Erosion Flow) *The* erosion flow *is defined as the vector field:*

$$\vec{\epsilon}_E(f)(x) = -x + \arg \inf_{y \in E} e_{y,x}(f(y)).$$

Note that the there need not be a unique point where the infimum is attained. In practice the flow is calculated by taking an arbitrary element of the set of minimizing vectors.

With the flow operator it is of course possible to calculate the erosion itself:

$$\epsilon(f)(x) = f(x + \vec{\epsilon}(f)(x)).$$

The flow operator is not new in morphology. The criterion calculation in the *value-and-criterion operator structure* of [3] is essentially the erosion flow operator. Also in the formulation of *h-adjunctions, Goutsias and Heijmans [12]* *use the erosion (and dilation) flow operators (although they do not explicitly specify them as flow field operators).*

In the theory of h-adjunctions an h-erosion is constructed as:

$$\epsilon(f)(x) = f(x + \vec{\epsilon}(h(f))(x)),$$

where h is a mapping from the *range* V of the image $f : E \to V$ to a new range V'. The usefulness of this theory lies in the fact that V need not be a complete lattice (and thus does not permit the construction of 'real' morphological operators). As long as there exists a semantically meaningful mapping from V onto a complete lattice V' we are able to define h-morphological operators. It is not morphology in the classical sense, but several important properties from the complete lattice framework carry over to the theory of h-morphology, including the construction of h-openings and h-closings.

It should be noted that in our use of morphological flow operators the image operators that are used are not range operators; they are spatial image operators. The h-morphology theory (and using the flow to define practical h-erosions and h-dilations) is therefore not directly applicable for our purposes. In this report we do not pursue this matter any further.

3. The generalized Kuwahara-Nagao operator

The Kuwahara-Nagao image operator is based on the ability to calculate mean and variance of local image patches.

Definition 2 (Mean and Variance Operators) *The mean $\mu_w(f)$ of a local image patch is calculated with a convolution*

$$\mu_w(f) = f * w,$$

where w is a smoothing convolution kernel. The local variance $\sigma(f)$ is also calculated using convolutions:

$$\sigma_w(f) = f^2 * w - (f * w)^2.$$

The classical Kuwahara-Nagao operator (see Fig. 2 for some examples) can then be formulated using the erosion flow defined in the previous section:

Definition 3 (Kuwahara-Nagao operator) *The classical Kuwahara-Nagao filter uses uniform square kernels of size ρ for the mean and variance calculations and the same neighborhood for to calculate the erosion flow. If we set u^ρ to be the uniform convolution kernel and ρU to be the square structuring element of the same size then:*

$$\kappa(f)(x) = \mu_{u^\rho}(f)(x + \vec{\epsilon}_{\rho U}(\sigma_{u^\rho}(f))(x)).$$

We can improve the results using a Gaussian convolution kernel instead of a uniform kernel. The *Gaussian kernel* is defined by:

$$g^\rho(x) = \frac{1}{(\rho\sqrt{2\pi})^D} e^{-\frac{\|x\|^2}{2\rho^2}},$$

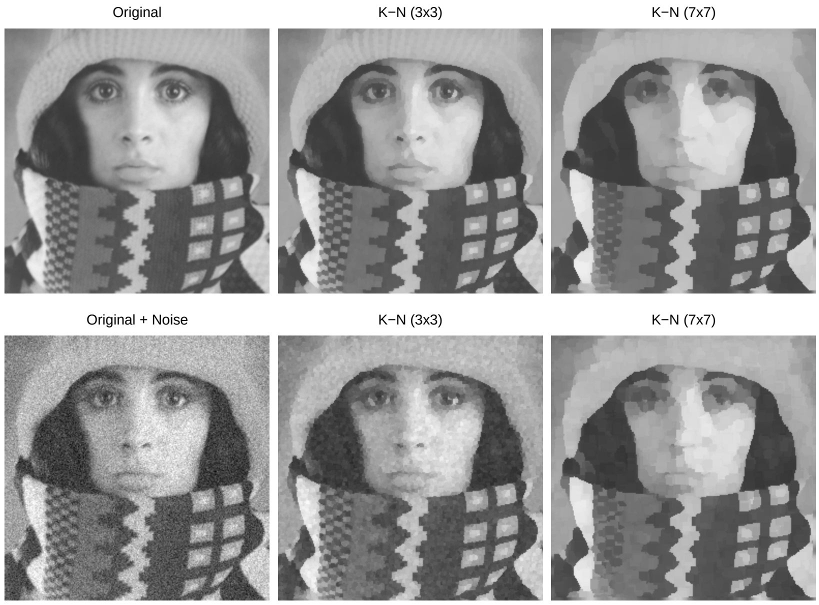

Figure 2. **Kuwahara-Nagao Operator.** On the first row from left to right: original image, K-N (3x3), K-N (7x7). On the second row the same layout but now we have added some noise to the original.

where D is the dimensionality of the image domain. The square structuring element used to calculate the erosion flow field, is replaced by a disk of radius ρ (i.e. the set ρB). This leads to:

Definition 4 (Gaussian Kuwahara-Nagao operator)

$$\kappa(f)(x) = \mu_{g^\rho}(f)(x + \vec{\epsilon}_{\rho B}(\sigma_{g^\rho}(f))(x)). \tag{1}$$

Comparing the results of the Gaussian K-N operator in Fig. 3 with the classical operator, we see that the Gaussian version shows more smooth results. The shape artifact as can be observed in the result of the classical K-N operator are less pronounces in the result of the Gaussian K-N operator.

4. A PDE formulation of the Kuwahara-Nagao operator

In order to link the G-K-N operator with modern edge preserving (and even edge *enhancing*) operators we will consider the case that the scale of smoothing is taken infinitesimally small. Iterating the resulting 'infinitesimal operator' results in an image evolution process that satisfies the G-K-N partial differential equation (PDE).

A first step in a PDE formulation of the Gaussian Kuwahara-Nagao operator is to look at $\vec{\epsilon}_{\rho B}$ for $\rho \to 0$.

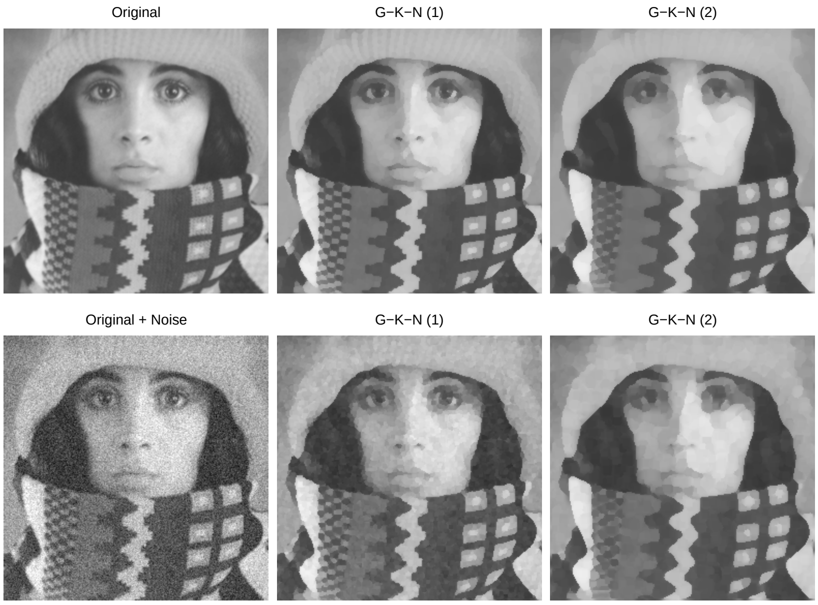

Figure 3. **Gaussian Kuwahara-Nagao Operator.** On the first row from left to right: original image, G-K-N (1), G-K-N (2) (the number in parentheses is the scale of the Gaussian kernel). On the second row the same layout but now we have added some noise to the original.

Proposition 5 *For $\rho \to 0$ and for almost everywhere differentiable images h we have:*

$$\vec{\epsilon}_{\rho B}(h) = -\rho \frac{\nabla h}{\|\nabla h\|}.$$

The proof is simple: a smooth function h locally can be approximated as a planar patch and within a disk of radius ρ such a function attains its maximal value on the boundary of the disk in the opposite gradient direction.

The next proposition that we need in the main theorem of this section is that for $\rho \to 0$ a Gaussian smoothing amounts to the addition of the Laplacian.

Proposition 6 *For $\rho \to 0$ we have:*

$$\mu_{g^{\sqrt{2\rho}}}(f) = f + \rho \nabla^2 f.$$

This is of course a well-known fact in linear scale-space theory: the Laplacian operator is the infinitesimal generator of the Gaussian scale-space. For the variance operator σ this leads to:

Proposition 7 *For $\rho \to 0$ we have:*

$$\sigma_{g^{\sqrt{2\rho}}}(f) = 2\rho \|\nabla f\|^2.$$

Proof: From the definition of the variance operator we have:

$$\begin{aligned}\sigma_{g^{\sqrt{2\rho}}}(f) &= f^2 * g^{\sqrt{2\rho}} - (f * g^{\sqrt{2\rho}})^2 \\ &= f^2 + \rho\nabla^2 f^2 - (f + \rho\nabla^2 f)^2 \\ &= \rho(\nabla^2 f^2 - 2f\nabla f - \rho(\nabla^2 f)^2).\end{aligned}$$

The last term in the above equation is quadratic in ρ and thus can be neglected in first order. Thus:

$$\begin{aligned}\sigma_{g^{\sqrt{2\rho}}}(f) &= \rho(\nabla^2 f^2 - 2f\nabla f) \\ &= \rho(2\|\nabla f\|^2 + 2f\nabla^2 f - 2f\nabla^2 f) \\ &= 2\rho\|\nabla f\|^2.\end{aligned}$$

■

Prop. 7 shows that the squared gradient norm is proportional to the local variance of the luminance observations.

Now we are in a position to look at κf (as defined in Eq. 1) for the case that $\rho \to 0$. We take the resulting operator as the 'infinitesimal generator' of a family of images $\mathcal{F}(x, \rho)$ with initial condition $\mathcal{F}(x, \rho = 0) = f(x)$. We slightly change the G-K-N operator to allow for different scales in the mean and variance operators on the one hand and in the erosion flow operator:

$$\kappa(f)(x) = \mu_{g^{\sqrt{2\rho}}}(f)(x + \vec{\epsilon}_{\lambda\rho B}(\sigma_{g^{\sqrt{2\rho}}}(f))(x)). \tag{2}$$

Here the λ positive scalar defines the balance between the scale of the linear operators and the scale of the erosion flow operator. The reason for this change will become clear later on. Also note that we have re-parameterized the scale of the Gaussian kernel to obtain the smoothing operator that is a linear additive semi group (just as the erosion $\epsilon_{\rho B}$ is a linear additive semi group under iterations).

Theorem 8 (Kuwahara-Nagao PDE) *Iterating the Gaussian Kuwahara-Nagao operator (Eq. 2) in the limit for $\rho \to 0$ results in a one-parameter family of images $\mathcal{F}(x, \rho)$ that satisfies the PDE:*

$$\frac{\partial \mathcal{F}}{\partial \rho} = \nabla^2 \mathcal{F} - \lambda \frac{\nabla \mathcal{F}^{\mathsf{T}} H_{\mathcal{F}} \nabla \mathcal{F}}{\|H_{\mathcal{F}} \nabla \mathcal{F}\|}. \tag{3}$$

Proof: From the definition of the G-K-N operator we have:

$$\kappa(f)(x) = \mu_{g^{\sqrt{2\rho}}}(f)(x + \vec{\epsilon}_{\lambda\rho B}(\sigma_{g^{\sqrt{2\rho}}}(f))(x)).$$

First we use Prop. 5 to write:

$$\kappa(f)(x) = \mu_{g^{\sqrt{2\rho}}}(f)(x - \lambda\rho \frac{\nabla(\sigma_{g^{\sqrt{2\rho}}}(f))(x)}{\|\nabla(\sigma_{g^{\sqrt{2\rho}}}(f))(x)\|}).$$

A first order Taylor expansion is allowed ($\rho \to 0$) and results in:

$$\kappa(f) = \mu_{g^{\rho}}(f) - \lambda\rho \frac{\nabla(\sigma_{g^{\sqrt{2\rho}}}(f)) \cdot \nabla(\mu_{g^{\sqrt{2\rho}}}(f))}{\|\nabla(\sigma_{g^{\sqrt{2\rho}}}(f))\|}.$$

From now on we omit the argument x as it is the same for all functions involved. Then we can use Prop. 6 and Prop. 7 and rewrite the above into:

$$\kappa(f) - f = \rho \left(\nabla^2 f - \lambda \frac{(\nabla f)^{\mathsf{T}} \nabla(\|\nabla f\|^2)}{\|\nabla(\|\nabla f\|^2)\|} \right),$$

where we have neglected the term in ρ^2. Note that $\nabla(\|\nabla f\|^2) = 2H_f \nabla f$ where H_f is the image Hessian. This finally leads to

$$\frac{\kappa(f) - f}{\rho} = \nabla^2 f - \lambda \frac{(\nabla f)^{\mathsf{T}} H_f (\nabla f)}{\|H_f \nabla f\|}.$$

The operator κ, in the limit $\rho \to 0$, when iterated defines a one parameter family of images $\mathcal{F}(\cdot, \rho)$ that satisfies the PDE:

$$\frac{\partial \mathcal{F}}{\partial \rho} = \nabla^2 \mathcal{F} - \lambda \frac{\nabla \mathcal{F}^{\mathsf{T}} H_{\mathcal{F}} \nabla \mathcal{F}}{\|H_{\mathcal{F}} \nabla \mathcal{F}\|}.$$

■

Eq. 3 shows that the G-K-N PDE is build from two parts. The first one, $\nabla^2 \mathcal{F}$, is easily interpreted as the linear diffusion term, i.e. a smoothing term. The second term has an edge sharpening effect. This is harder to interpret from the algebraic form given in theorem 8. The sharpening term is easier to interpret in case we rewrite it in terms of the gradient gauge. Subscript w traditionally denotes a derivative in the direction of the gradient and v is the derivative in the perpendicular direction.

Corollary 9 *In the gradient gauge the G-K-N PDE is given by:*

$$\frac{\partial \mathcal{F}}{\partial \rho} = \nabla^2 \mathcal{F} - \lambda \frac{\operatorname{sign}(\mathcal{F}_{ww})}{\sqrt{1 + \frac{\mathcal{F}_{vw}^2}{\mathcal{F}_{ww}^2}}} \mathcal{F}_w.$$

The proof is simple when we realize that $\mathcal{F}_v = 0$. We can go one (qualitative) step further and rewrite the above expression using the fact that the *flow line curvature* is given by $c_f = -\mathcal{F}_{vw}/\mathcal{F}_w$. Then in those places in the image of zero flow line curvature we arrive at:

Corollary 10 *In the gradient gauge and for the locations with low (zero) flow line curvature, the G-K-N PDE is given by:*

$$\frac{\partial \mathcal{F}}{\partial \rho} = \nabla^2 \mathcal{F} - \lambda \operatorname{sign}(\mathcal{F}_{ww}) \mathcal{F}_w. \tag{4}$$

Now the term $-\operatorname{sign}(\mathcal{F}_{ww})\mathcal{F}_w$ can be recognized as a edge enhancing term. This term on its own (i.e. without the smoothing term) is well-known for its sharpening properties. The Kramer (see [13]) morphological sharpening operator is the finite size discrete implementation of this sharpening PDE.

Discretizing the diffusion term in the PDE in Eq. 4 is easy. The simplest finite difference scheme is:

$$(\nabla^2 \mathcal{F})_{i,j} = \mathcal{F}_{i,j+1} + \mathcal{F}_{i-1,j} - 4\mathcal{F}_{i,j} + \mathcal{F}_{i+1,j} + \mathcal{F}_{i,j-1}.$$

Discretization of the sharpening term is more difficult. In classical numerical schemes to solve these types of PDE's the *up-wind* schemes are used. In [14] we have shown that the term $-\text{sign}\,(\mathcal{F}_{ww})\,\mathcal{F}_w$ corresponds with an erosion in case $\mathcal{F}_{ww} > 0$ and a dilation in case $\mathcal{F}_{ww} < 0$. Sub-pixel erosions and dilations based on an interpolation within a 3×3 neighborhood (using only the 4 connected neighbors of the central pixel) are given in [14]. For the dilation a simple scheme is:

$$(+\mathcal{F}_w)_{i,j} = \sqrt{(D^{-x}_{i,j}\mathcal{F} \vee D^{+x}_{i,j}\mathcal{F})^2 + (D^{-y}_{i,j}\mathcal{F} \vee D^{+y}_{i,j}\mathcal{F})^2},$$

where

$$D^{-x}_{i,j}\mathcal{F} = 0 \vee \frac{\mathcal{F}_{i-1,j} - \mathcal{F}_{i,j}}{\Delta x}, \qquad D^{+x}_{i,j}\mathcal{F} = 0 \vee \frac{\mathcal{F}_{i+1,j} - \mathcal{F}_{i,j}}{\Delta x}.$$

For the y-direction the schemes are equivalent. The choice of one of the above schemes is on a per pixel basis depending on the sign of $\mathcal{F}_{ww}$.

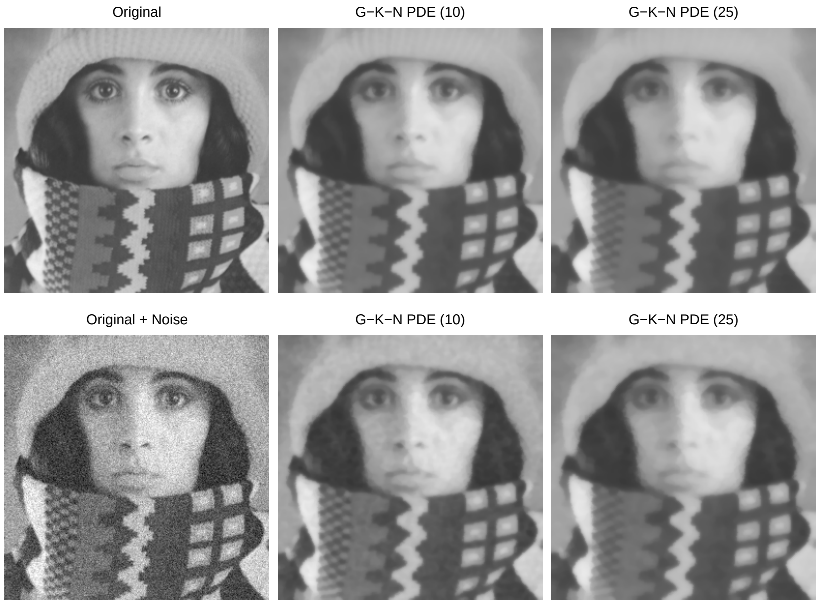

Figure 4. **Kuwahara-Nagao PDE Operator.** On the first row from left to right: original image, G-K-N PDE (10), G-K-N PDE (25). The value between brackets is the number of iterations. On the second row the same layout but now we have added some noise to the original.

In Fig. 4 numerical solutions of the G-K-N PDE (Eq. 4) are shown.

5. Conclusions

In this report we have shown that the Kuwahara-Nagao image operator can be cast in a PDE controlled image evolution process. The resulting PDE combines the linear diffusion (i.e. Gaussian smoothing) with morphological sharpening.

The results of the classical K-N operator and the newly formulated PDE evolution are qualitatively comparable. The main differences are:

- The PDE results are smoother than the classical implementations based on finite size neighborhoods.
- The PDE operator tends to preserve small details better the classical implementation.

The analysis in this report sheds some new light on an old and trusted tool in image processing. The Kuwahara-Nagao operator is known as an edge preserving smoothing operator. This report shows that we don't need to trust on our intuition to understand this claim but that it follows from basic principles, i.e. the combination of linear diffusion and morphological sharpening.

Acknowledgments

I would like to thank Joost van de Weijer who did some careful last minute proofreading. Any remaining errors are of course my own.

References

[1] M. Kuwahara, K. Hachimura, S. Eiho, and M. Kinoshita, Processing of RI-angiocardiographic images, in Digital Processing of Biomedical Images, K. Preston, Jr. and M. Onoe, Editors. New York: Plenum, 1976. pp. 187-202.

[2] M. Nagao and T. Matsuyama, "Edge preserving smoothing", Computer Graphics and Image Processing, vol. 9, no. 4, pp. 394–407, Apr. 1979.

[3] M.A. Schulze and J.A. Pearce, A Morphology-Based Filter Structure For Edge-Enhancing Smoothing, ICIP-94, Vol. 94, pp. 530–534

[4] D. Gabor, Information theory in electron microscopy, Laboratory Investigation, Vol. 14, No. 6, pp. 801–807, 1965

[5] M. Lindenbaum, M. Fischer and A.M. Bruckstein, On Gabor's contribution to image enhancement, Pattern Recognition, Vol. 27, No. 1, pp. 1–8

[6] J. Weickert, Anisotropic diffusion in image processing, Teubner Verlag, Stuttgart, 1997

[7] M. J. Black, G. Sapiro, D. H. Marimont, and D. Heeger, Robust Anisotropic Diffusion, IEEE Trans. Image Processing, 7:421-432, 1998.

[8] G. Sapiro, Geometric Partial Differential Equations and Image Analysis, Cambridge University Press, Cambridge-UK, 2001.

[9] P. Perona and J. Malik, Scale-space and edge detection using anisotropic diffusion, IEEE Trans. Pattern Anal. Machine Intell., vol. 12, pp. 629–639, July 1990.

[10] R. Kimmel, R. Malladi, and N. Sochen. Images as Embedded Maps and Minimal Surfaces: Movies, Color, Texture, and Volumetric Medical Images. IJCV, 39(2):111-129, Sept. 2000

[11] H.J.A.M. Heijmans. Morphological Image Operators. Academic Press, Boston, Mass., 1994.

[12] J. Goutsias, H. Heijmans, and K. Sivakumar, "Morphological operators for image sequences", Computer Vision and Image Understanding 62(3), pp. 326–346, 1995

[13] J.G.M. Schavemaker, M.J.T. Reinders, and R. van den Boomgaard. Image sharpening by morphological filtering. In Proc. of the IEEE/EURASIP Workshop on Nonlinear Signal and Image Processing. Mackinac Island, Michigan USA, September 1997.

[14] R. van den Boomgaard. Numerical solutions of morphological partial differential equations. In Scale-Space '99. Kluwer Academic Publishers, 1999.