MORPHOLOGICAL SCALE-SPACES, SCALE-INVARIANCE, AND LIE GROUPS

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Abstract This paper is concerned with the axiomatics of image scale-spaces. A major ingredient is the invariance under a family of scalings. A scaling can be defined explicitly, but also through a system of ODE's. It is shown how existing PDE approaches for scale-spaces can be fit into the axiomatic framework, either directly or with the help of Lie group methods.

Keywords: Scale-space, scale invariance, directed scale-space, PDE, Hamilton-Jacobi equation, Lie group, vector field.

1. Introduction

Recently we have proposed a new axiomatic definition of scale-spaces [8, 18, 5]. In this definition, the construction of a scale-space comprises three consecutive steps: (*i*) down-scale the image, (*ii*) apply a given image operator at unit scale, and (*iii*) resize the image to its original scale. In this paper we extend our approach with some new results:

- We show that every scale-space is *directed*. This means, somewhat imprecisely, that it decreases the information content of an image over time.
- We explain, by means of a direct method as well as with the aid of Lie groups, that the PDE approach towards scale-spaces is compatible with our definition.
- We give some new examples, both of scalings and of scale-spaces.

We give a brief overview of the paper. First we recall the axiomatic definition of a scale-space in Section 2. In Section 3 we prove that every scale-space is directed. In Section 4 and 5 we devote special attention to morphological scale-spaces. As said before, our definition is consistent with the PDE approach. In Section 6 we use a direct method to show this consistency and in Section 7 we present an alternative, but very powerful, technique based on Lie groups. Until this point, the scaling families considered in this paper belong to a simple two-parameter class. In Section 8 we show that this class can be extended

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considerably by using ODE's for their definition. We conclude with some final remarks in Section 9.

2. Axiomatic definition of scale-spaces

In [8] we have proposed an axiomatic definition of a scale-space on an image space $\mathcal{L}$; see also [18, 5]. This definition comprises the following ingredients:

(i) a one-parameter family $\{S(t)\}_{t>0}$ of *scaling operators* on $\mathcal{L}$;

(ii) an operator ψ on $\mathcal{L}$, called *kernel operator*, which describes the action at a given unit scale.

A one-parameter family $S = \{S(t)\}_{t>0}$ of operators on $\mathcal{L}$ is called a *scaling* if

$$S(1) = \mathsf{id} \text{ and } S(t)S(s) = S(ts), \quad s,t > 0. \tag{1}$$

Thus S comprises a commutative group with $S(t)^{-1} = S(1/t)$ for $t > 0$. For the time being, take $\mathcal{L}$ to be the set of functions mapping $\mathbb{R}^d$ into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. An important family of scalings is the family $S^{p,q}$, with $p, q \in \mathbb{R}$, given by

$$S^{p,q}(t)f = t^q f(\cdot/t^p), \quad t > 0, \tag{2}$$

for $f \in \mathcal{L}$. Of particular interest are the following scalings:

– *spatial scaling*: $p = 1,\ q = 0$: a function f is scaled in the spatial domain but not in the gray-level domain;

– *quadratic scaling*: $p = 1/2,\ q = 0$;

– *umbral scaling*: $p = 1,\ q = 1$: the corresponding scaling $S^{1,1}$ scales the region in $\mathbb{R}^d \times \overline{\mathbb{R}}$ beneath the graph of f.

In Section 8 we will discuss a more general family of scalings.

Definition 1 *Let S be a scaling on $\mathcal{L}$. The family $\{T(t)\}_{t>0}$ of operators on $\mathcal{L}$ is called an* $(S,+)$-scale-space *if*

$$\begin{aligned} T(t)S(t) &= S(t)T(1),\ t > 0 &(3)\\ T(t)T(s) &= T(t+s),\ s,t > 0. &(4) \end{aligned}$$

It is called an $(S,\vee)$-scale-space *if, instead of* (4)*, the following holds:*

$$T(t)T(s) = T(t \vee s),\ s,t > 0. \tag{5}$$

Scale-spaces satisfying $T(t)T(s) = T(t+s)$ are called *additive*, whereas they are called *supremal* if $T(t)T(s) = T(t \vee s)$. If we do not want to be explicit about the type of scale-space, we shall write $T(t)T(s) = T(t \dot{+} s)$, where '$\dot{+}$' can mean '+' as well as '$\vee$'. If $\{T(t)\}_{t>0}$ is a scale-space, then the operator $\psi = T(1)$ is called the *kernel operator*. From (3) we conclude that

$$T(t) = S(t)\psi S(t)^{-1}, \tag{6}$$

and sometimes we write $T_\psi(t)$ instead of $T(t)$ to make the dependence on ψ explicit. Furthermore, we can easily establish that $T(t)S(s) = S(s)T(t/s)$ for $t, s > 0$. This yields in particular that

$$S(t)T(r)S(t)^{-1} = T(rt), \; r, t > 0\,. \tag{7}$$

Therefore, $T_\psi(r)$ is a kernel operator if and only if ψ is one (with respect to the same scaling family). In other words, the choice of the unit scale is irrelevant.

3. Directionality

In this section we show that scale-spaces cannot exhibit any kind of behaviour if time increases. In this respect, we refer to the monograph of Weickert [20], where the author characterises a class of Lyapunov functions for a general nonlinear diffusion process. The existence of a Lyapunov function for a scale-space $T(t)$ guarantees the simplification of the one-parameter family of images $u(t) = T(t)f$ as time increases. Our framework is too general to prove the existence of a Lyapunov function, but we are nevertheless able to prove that the collection $\{u(t) \mid t > 0\}$ is totally ordered.

We first recall some results concerning idempotence of the kernel operator and the scale-space operators; see [8] for a proof.

Proposition 2 *Let $T(t)$ be given by* (6)*, where ψ, S are an operator and a scaling on $\mathcal{L}$, respectively.*
(*a*) *If ψ is idempotent then $T(t)$ is idempotent too, for every $t > 0$ (but not necessarily a scale-space).*
(*b*) *If $\{T(t)\}_{t>0}$ is a supremal scale-space, then every operator $T(t)$ (including $T(1) = \psi$) is idempotent.*
(*c*) *If ψ is idempotent and $\{T(t)\}_{t>0}$ is an additive scale-space, then $T(t) = \psi$ for every $t > 0$.*

In [8] we have formulated the following open problem: "can every $(S, \dot{+})$-scale-space be endowed with a total ordering which is compatible with the ordering of the positive reals." In other words, we are questioning the existence of a total ordering $\preceq$ on the set of operators $\{T(t) \mid t > 0\}$ such that

$$T(s) \preceq T(t) \text{ if } s \leq t\,.$$

If such an ordering exists, then one might conclude that the information content decreases with time. We say that the scale-space is *directed*. Directionality excludes any sort of periodic behaviour.

Proposition 3 *Let $\{T(t)\}_{t>0}$ be an $(S, \dot{+})$-scale-space with kernel ψ. If there exist $t_1, t_2 > 0$ with $t_1 \neq t_2$ and $T(t_1) = T(t_2)$ then $T(t) = \psi$ for every $t > 0$ (and hence ψ is idempotent).*

Proof 4 *Consider first the case that $\{T(t)\}_{t>0}$ is a supremal scale-space. If $T(t_1) = T(t_2)$, with $t_1 < t_2$ and if $s \in [t_1, t_2]$, then*

$$T(s) = T(t_1 \vee s) = T(t_1)T(s) = T(t_2)T(s) = T(t_2 \vee s) = T(t_2)\,.$$

Using (7) *we get that for every fixed* $s > 0$*,* $T(st)$ *is constant for* $t \in [t_1, t_2]$*. Obviously, this is possible only if* $T(t)$ *is constant for all* $t > 0$*.*

Next, assume that $\{T(t)\}_{t>0}$ *is an additive scale-space. Assume that* $T(t) = T(t+p)$ *for some* $t, p > 0$*. Choose an integer* $k > 1$ *such that* $kp > t$*. First,* $T(t+p) = T(t)$ *implies* $T(t+kp) = T(t)$*, hence*

$$T(2kp) = T(t+kp)T(kp-t) = T(t)T(kp-t) = T(kp)\,,$$

which can also be written as $T(kp)T(kp) = T(kp)$*. Therefore*

$$S(kp)\psi^2 S(kp)^{-1} = S(kp)\psi S(kp)^{-1}\,,$$

hence $\psi^2 = \psi$*. From Proposition 2(c) it follows that* $T(t) = \psi$ *for* $t > 0$*.*

This result shows that there are two possibilities: either $T(t)$ does not depend on t or $T(s) \neq T(t)$ if $s \neq t$. In both cases we can easily conclude that $T(t)$ is directed; in the latter case the total ordering of the positive reals induces a total ordering on the set of operators $\{T(t) \mid t > 0\}$.

Corollary 1 *Every* $(S, \dot{+})$*-scale-space is directed.*

Suppose we are given two images $f_1 = T(t_1)f$ and $f_2 = T(t_2)f$ where t_1, t_2 and f are unknown. If the scale-space T(t) is directed and non-constant, then we can check in principle whether t_1 is smaller or larger than t_2. For, if there exists an $s > 0$ such that $f_2 = T(s)f_1$ then t_2 is larger, otherwise it is smaller. In practice, however, we would like to have some computational mechanism, like the aforementioned Lyapunov function, to check this.

4. Morphological scale-spaces

In [8] we have discussed in great detail the case where the kernel operator ψ is a morphological operator. We briefly recall the most important results here.

Proposition 5 *Let* $\mathcal{L}$ *be a complete lattice and assume that every scaling operator* $S(t)$ *is increasing. If* (ε, δ) *is an adjunction on* $\mathcal{L}$*, then* $(T_\varepsilon(t), T_\delta(t))$ *is an adjunction, too, for every* $t > 0$*. Moreover, if* $T_\varepsilon(t)$ *defines an* $(S, \dot{+})$*-scale-space on* $\mathcal{L}$*, then* $T_\delta(t)$ *defines an* $(S, \dot{+})$*-scale-space as well. Finally, the opening* $\delta\varepsilon$ *induces an* $(S, \vee)$*-scale-space and* $T_{\delta\varepsilon}(t) = T_\delta(t)T_\varepsilon(t)$*. (Similar for closings.)*

Assume that $\mathcal{L} = \mathrm{Fun}(\mathbb{R}^d)$ and consider the family of erosions given by the so-called *infimal convolution*:

$$\varepsilon_b(f)(x) = (f \mathbin{\text{fi}} b)(x) = \bigwedge_{h \in \mathbb{R}^d} [f(x-h) + b(h)]\,. \tag{8}$$

Here b is called the *structuring function*. Often, we will omit the subscript b and write ε rather than ε_b. Henceforth we assume that the function b is lower semi-continuous and convex, and satisfies $b(x) > -\infty$ for every $x \in \mathbb{R}^d$. Taking $S = S^{p,q}$, one finds that $T_\varepsilon(t)$ equals

$$T_\varepsilon(t)f = f \mathbin{\text{fi}} S(t)b\,. \tag{9}$$

A convex function b is called *subpolynomial of degree k*, where $k \geq 1$, if

$$b(tx) = t^k b(x), \quad x \in \mathbb{R}^d,\ t > 0\,.$$

b is called subpolynomial of degree $+\infty$ if it is an indicator function.

Proposition 6 *The family $\{T_\varepsilon(t)\}_{t>0}$ given by* (9) *defines an $(S^{p,q}, +)$-scale-space in each of the following two cases:*

(*a*) $p = q = 1$*;*

(*b*) $q < p < 1$ *or* $1 < p < q$ *and b is subpolynomial of degree k with* $k = (1-q)/(1-p)$.

The equation $k = (1-q)/(1-p)$ does not always have a unique solution for a given k. This suggests that more than one scale-space can be associated with a structuring function b, but it is easy to see that they all coincide. Putting

$$u(t, \cdot) = f \text{ fi } S(t)b\,, \tag{10}$$

where $S = S^{1,1}$, it is known [7] that u is a solution of the Hamilton-Jacobi equation

$$u_t = -H(\nabla u)\,, \tag{11}$$

where the *Hamiltonian* H is the *Young-Fenchel conjugate* of b:

$$H(x) = b^\star(x) = \bigvee_{y \in \mathbb{R}^d} [\, \langle x, y \rangle - b(y)\,]\,. \tag{12}$$

The structuring function b is called the *Lagrangian.* Since $(b^\star)^\star = b$, it follows that b can be recovered from H through $b = H^\star$. The solution formula in (10) is sometimes called the *Lax-Oleinik formula.* If H is convex and *coercive*, then $u(t, \cdot)$ given by (10) is a so-called *viscosity solution* of (11); refer to [3] for more details.

In [8] we have also given a complete characterisation of scale-spaces induced by structural openings. From Proposition 2 we know that such scale-spaces are necessarily supremal, unless they are time-independent in which case they are both additive and supremal (with a kernel which is idempotent).

5. Self-dual erosion scale-space

In [10], Keshet has introduced a class of self-dual morphological operators. These have been investigated in a more systematic manner in [6]. The basic concept underlying this family of erosions is a self-dual partial ordering on the reals:

$$s \preceq t \text{ if } t \leq s \leq 0 \text{ or } 0 \leq s \leq t\,. \tag{13}$$

With this partial ordering, the set of reals becomes a *complete inf-semilattice* (or *cisl*) where the infimum but not the supremum exists. With any flat structuring element $B \subseteq \mathbb{R}^d$ one can associate an *erosion* ε on the cisl comprising all functions from $\mathbb{R}^d$ into $\mathbb{R}$, provided with the pointwise ordering of $\mathbb{R}$ in (13):

$$\varepsilon(f)(x) = \curlywedge_{b \in B} f(x - b)\,.$$

Here 'f' denotes the infimum associated with the partial ordering '$\preceq$'. Choosing the spatial scaling $(S(t)f)(x) = f(x/t)$ we arrive at the family $\{T_\varepsilon(t)\}_{t>0}$ given by

$$(T_\varepsilon(t)f)(x) = \curlywedge_{b\in B} f(x - tb)\,.$$

Assuming that B is a closed set, it is easy to verify that $\{T_\varepsilon(t)\}_{t>0}$ defines an additive scale-space if and only if B is convex. If we take for B the closed unit ball, then we can show that $u(t,x) = (T_\varepsilon(t)f)(x)$ satisfies the PDE

$$u_t = -\mathrm{sign}\,(u)\|\nabla u(t,x)\|\,,$$

where $\mathrm{sign}\,(u)$ is the sign of u and $\|\cdot\|$ is the Euclidean norm. A comprehensive discussion of this PDE can be found in [15]. In [12], Meyer and Maragos derive a similar PDE modeling morphological levelings.

6. Scale-spaces and PDE's

Our approach towards scale-spaces is an algebraic one and as such it deviates from the mainstream of the literature on scale-spaces which deals with the PDE approach. In this section we show that the two approaches are less different than might appear at first sight.

In the previous section we have seen that the erosion scale-space can be described by a PDE of Hamilton-Jacobi type. An even simpler, linear PDE is the heat equation

$$u_t = \Delta u\,, \tag{14}$$

describing the well-known Gaussian scale-space [21, 11]. In fact we have shown in [8] that this scale-space fits well into axiomatic framework if one chooses the convolution with a Gaussian as the kernel operator in combination with the quadratic scaling. In the literature one can find numerous other PDE's that are exploited in image processing [4, 5, 20]. Alvarez, Guichard, Lions and Morel [1, 2] derived a family of nonlinear PDE models under the assumption that the multiscale analysis under consideration is causal and regular:

$$u_t(t,x) = F(\nabla^2 u(t,x), \nabla u(t,x), u(t,x), x) \quad \text{and} \quad u(0,\cdot) = f\,. \tag{15}$$

Here F is a function mapping $M \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}$, with M being the space of symmetric matrices, which is nondecreasing in its first argument with respect to the partial ordering of M.[†] Alvarez *et al.* [1] have also shown that one arrives at

$$u_t = \nabla u\left(\mathrm{curv}(u)\right)^{\frac{1}{3}}\,, \tag{16}$$

if one assumes in addition that the multiscale analysis is invariant under changes of contrast and affine transformations. Here $\mathrm{curv}(u) = \mathrm{div}(\nabla u/\|\nabla u\|)$ is the curvature of the level line of $u(t,x)$ passing through x. They refer to this system as the *affine morphological scale-space* (*AMSS*) model; see also [16].

[†] In fact, the original equation given by Alvarez *et al.* also contains the argument t; we have suppressed it here since we are only interested in time-homogeneous equations.

In the sequel we assume that we can associate a solution $u(t) = T(t)f$ with the initial value problem given by (15). As the system is time-homogeneous, the semigroup condition $T(t+s) = T(t)T(s)$ holds. In [5] we have shown how to formulate the scaling-invariance condition in (3) in terms of the underlying equation. As before, we shall restrict ourselves to scalings $S^{p,q}$. Assume that $u(t,x)$ is a solution of (15), fix $s > 0$, and define

$$\tilde{u}(t,x) = s^q u(t/s, x/s^p)\,. \tag{17}$$

In order for (3) to hold, it is necessary and sufficient that $\tilde{u}$ is a solution of (15) with f replaced by $S(s)f$. This amounts to the following condition on F:

$$s^{q-1}F(w,v,u,x) = F(s^{q-2p}w, s^{q-p}v, s^q u, s^p x)\,. \tag{18}$$

For example, consider the Hamilton-Jacobi equation (11). In this case, $F(w,v,u,x) = -H(v)$ and (18) yields

$$s^{q-1}H(w) = H(s^{q-p}w)\,.$$

If H is subpolynomial of degree ℓ, then the right hand-side equals $s^{\ell(q-p)}H(w)$ and we find $q-1 = \ell(q-p)$. These values of p,q,ℓ correspond with those in Proposition 6 with $b = H^*$. Note that H is subpolynomial of degree ℓ if and only if $b = H^*$ is subpolynomial of degree k where $1/\ell + 1/k = 1$.

7. Scale-invariance and Lie groups

Lie groups provide a powerful method to describe and analyse invariance properties of differential equations. In fact, it can be shown that the invariance of a PDE under a certain group of transformations, which usually gives rise to nonlinear conditions, can be replaced by much simpler linear conditions governing a sort of infinitesimal invariance. This requires the computation of the infinitesimal generator(s) of the symmetry group. In this section we outline this approach. We adopt the notation and terminology from Olver's outstanding monograph [13].

Given a smooth function $u = f(x)$ with $f : \mathbb{R}^p \to \mathbb{R}$ involving p independent variables $x = (x^1, \ldots, x^p)$. For an unordered k-tuple $J = (j_1, \ldots, j_k)$ with $1 \le j_i \le p$ we define

$$\partial_J f(x) = \frac{\partial^k f(x)}{\partial_{x^{j_1}}\partial_{x^{j_2}}\cdots\partial_{x^{j_k}}}\,.$$

We call J an *unordered multi-index of order* k (i.e., $k = \mathrm{card}(J)$). Note that there exist $p_k = \binom{p+k-1}{k}$ of such k'th order derivatives. The product space $U^{(n)} = \mathbb{R}^p \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \times \cdots \times \mathbb{R}^{p_n}$ can represent all derivatives of a function $u = f(x)$ of all orders between 0 and n. Given $u = f(x)$, there is an induced function $u^{(n)} = \mathrm{pr}^{(n)} f(x) \in U^{(n)}$, called the *n'th prolongation of* f, which is given by the equations

$$u_J = \partial_J f(x)\,.$$

In other words, $\mathrm{pr}^{(n)} f(x)$ represents the values of f and all derivatives up to order n at x. For example, if $u = u(t,x)$ with $t, x \in \mathbb{R}$, then $u^{(2)} = (u; u_t, u_x; u_{tt}, u_{tx}, u_{xx})$.

An n'th order PDE describing a smooth function u involving p independent variables can be written as

$$\Delta_\nu(x, u^{(n)}) = 0,\ \nu = 1, \ldots, l\,, \tag{19}$$

and a solution is a smooth function $u = f(x)$ such that

$$\Delta_\nu(x, \mathrm{pr}^{(n)} f(x)) = 0,\ \nu = 1, \ldots, l\,.$$

A symmetry group of this system is a (local) group G of transformations acting on a submanifold of $X \times U$, which transforms solutions of the system to other solutions. The symmetry of the system can be reduced to the more tractable problem of establishing the invariance of the variety given by (19) under the so-called prolonged group action $\mathrm{pr}^{(n)}G$; see [13, Thm 2.27] for a precise formulation. Moreover, this invariance can be formulated in terms of the prolongations of the infinitesimal generators (or *vector fields*) of the group G.

In the previous section we have seen that the system $u_t = A(u)$ models an $(S^{p,q}, +)$-scale-space if the one-parameter transformation group

$$(x, t, u) \mapsto (s^p x, st, s^q u) \tag{20}$$

is a symmetry group. The associated vector field is

$$\mathbf{v} = px\partial_x + t\partial_t + qu\partial_u\,. \tag{21}$$

To see this, solve the ODE system $\frac{d}{d\theta}(X, T, U) = (pX, T, qU)$ with initial condition $(X(0), T(0), U(0)) = (x, t, u)$ and substitute $s = \exp(\theta)$. In the above exposition we have assumed for simplicity that the spatial dimension is one. In case x is higher dimensional, i.e., $x = (x^1, \ldots, x^d)$, we have to replace $x\partial_x$ by $x^1\partial_{x^1} + \cdots + x^d\partial_{x^d}$.

Throughout this paper we restrict ourselves to PDE's of order $n \leq 2$, whence it follows that we only need to compute the first and second prolongation of $\mathbf{v}$. A straightforward computation, following the exposition of [13, Section 2.3] shows that

$$\mathrm{pr}^{(1)}\mathbf{v} = \mathbf{v} + (q-p)u_x \frac{\partial}{\partial u_x} + (q-1)u_t \frac{\partial}{\partial u_t} \tag{22}$$

$$\mathrm{pr}^{(2)}\mathbf{v} = \mathrm{pr}^{(1)}\mathbf{v} + (q-2p)u_{xx} \frac{\partial}{\partial u_{xx}}\,. \tag{23}$$

We give two examples.

Example 1 (Heat equation) Applying $\mathrm{pr}^{(2)}\mathbf{v}$ to the heat equation $u_t = u_{xx}$, which is of second order, we find

$$(q-1)u_t = (q-2p)u_{xx}\,.$$

Substituting $u_t = u_{xx}$ yields the solution $p = \frac{1}{2}$. This means that the associated Gaussian scale-space is an $(S^{\frac{1}{2},q}, +)$-scale-space for every q. Note that, due to the linearity of the system, q can be chosen arbitrary.

Example 2 (Hamilton-Jacobi equation) The Hamilton-Jacobi equation $u_t = -H(u_x)$, which we have derived in (11), is a first-order equation. Applying $\mathrm{pr}^{(1)}\mathbf{v}$ yields (assuming that H is smooth):

$$(q-1)u_t = -(q-p)u_x H'(u_x)\,.$$

Substituting $u_t = -H(u_x)$ and replacing u_x by v we get

$$(q-1)H(v) = (q-p)vH'(v)\,.$$

This equation is solved without any further assumption on H if $p = q = 1$, which is consistent with the result in Proposition 6(a). Another solution associated with $p = \frac{1}{2}, q = 0$ is $H(v) = v^2/2$. The corresponding scale-space is called *parabolic scale-space*; see [9, 17, 19].

8. Other scalings

The key observation in the previous section is that the scaling $S^{p,q}$ can be associated with the vector field in (21).† We replace this vector field with the more general one

$$\mathbf{v} = \xi(x)\partial_x + t\partial_t + \varphi(u)\partial_u\,, \tag{24}$$

for given functions ξ, φ. Let $X(\theta, x)$ be the solution of the initial value problem $\frac{dX}{d\theta} = \xi(X),\ X(0) = x$ (similar definition of $U(\theta, u)$). Furthermore, let X^{-1} be the inverse of X, i.e., $X^{-1}(\theta, y)$ is the unique solution of $X(\theta, x) = y$. Then S given by

$$(S(t)f)(x) = U\big(\log t, f(X^{-1}(\log t, x))\big)\,, \tag{25}$$

defines a scaling. Moreover, $s(t,x) = (S(e^t)f)(x) = U\big(t, f(X^{-1}(t,x))\big)$ is the solution of the initial value problem

$$s_t + \xi(x)s_x = \varphi(s),\ \ s(0,x) = f(x)\,. \tag{26}$$

To check if the semigroup $\{T(t)\}_{t>0}$ associated with the system $u_t = A(u)$ is an $(S,+)$-scale-space we may compute the n'th prolongation of the vector field $\mathbf{v}$ in (24), where n is the highest order derivative occurring in A.

Let us consider as an example the Hamilton-Jacobi equation $u_t = -H(u_x)$. The 1'st order prolongation of $\mathbf{v}$ in (24) is

$$\mathrm{pr}^{(1)}\mathbf{v} = \mathbf{v} + (\varphi'(u) - \xi'(x))u_x\frac{\partial}{\partial u_x} + (\varphi'(u) - 1)u_t\frac{\partial}{\partial u_t}\,. \tag{27}$$

Note that for $\xi(x) = px$ and $\varphi(u) = qu$ this coincides with (22). Application to $u_t = -H(u_x)$ and substitution of $v = u_x$ yields

$$(\varphi'(u) - 1)H(v) = (\varphi'(u) - \xi'(x))vH'(v)\,.$$

An interesting solution is found if we assume that $\xi'(x) = 1$ and $H(v) = vH'(v)$. This latter ODE is solved e.g. by the Hamiltonian $H(v) = |v|$. The

†For simplicity we restrict ourselves to the 1D case here.

corresponding Lagrangian $b = H^*$ is the indicator function associated with the unit interval $[-1, 1]$. The solution of the corresponding PDE $u_t = -|u_x|$ is given by the flat erosion of the input function with structuring element $[-t, t]$. This generates an $(S, +)$-scale-space for every scaling of the form (25) with $X(\theta, x) = \exp(\theta)x$, that is

$$(S(t)f)(x) = U(\log t, f(x/t)),$$

where $U(\theta, u)$ is the solution of $\frac{dU}{d\theta} = \varphi(U)$ with $U(0, u) = u$.

9. Conclusions

We have presented a general axiomatic definition of scale-spaces which does not use PDE's but which is entirely algebraic. This is interesting, as such a definition allows scale-spaces which cannot be described by a PDE. For supremal scale-spaces, this may not be too surprising, but there do also exist additive scale-spaces, such as the linear scale-space using the Cauchy kernel, which cannot be modeled by a PDE; see [14] for more details.

Fortunately, our definition is consistent with the PDE approach advocated by various researchers. We have given two different methods to prove this consistency, a direct method and a method based on Lie groups.

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